QUANTUM FIELD THEORY

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Contents

Chapter 1

Field and Lorentz Transformation

1.1 Fields

The field $\phi(\mathbf{x})$ is a set of dynamical degrees of freedom distributed all over the points $\mathbf{x}$ in the three-dimensional space. There are varieties of field in Nature. The familiar examples may be an electric field $\mathbf{E}(\mathbf{x})$ and a magnetic field $\mathbf{B}(\mathbf{x})$. The field $\phi(\mathbf{x})$ may change its value in accordance with the time variation of its surroundings. What is more, the field itself evolves its motion by its own force. Its time evolution $\phi(\mathbf{x}, t)$ is governed by the equation of motion of the field $\phi(\mathbf{x})$.

One of the basic facts recognized at the microscopic level of Nature is that

Any Dynamical Freedom Reveals Its Dynamics as Quantum Dynamics,

where the dynamical freedom behaves as a quantum mechanical operator. The field $\phi(\mathbf{x})$ behaves as a quantum field operator $\hat{\phi}(\mathbf{x})$. The astonishing consequence of this "Quantum Principle" is that the quantum field $\hat{\phi}(\mathbf{x})$ describes, through its quantum excitation, a quantum mechanical particle specific to the field $\phi(\mathbf{x})$. Any of the quantum mechanical particles of species "s" is described by its own field $\phi_{\mathrm{s}}(\mathbf{x})$. The quantum electro-magnetic field describes a particle "photon", which reveals itself at a macroscopic level as a light.

One more basic fact in Nature is summarized as the "Principle of Relativity". The principle asserts that

The Law of Nature is Equally Realized in Any Inertial System.

For the field $\phi(\mathbf{x})$, this assertion amounts to that the equation of motion of the field takes the same form in any inertial system. Precisely speaking, the form of the equation of motion written down in terms of the time t and the coordinate $\mathbf{x}$ of one inertial system is identical to the form written down in termes of the time t' and the coordinate $\mathbf{x}'$ of another inertial system. If the form of two expressions were different, we might be able to distinguish one inertial system from another inertial system by carrying out the same experiment in each inertial system. The clear experiment may be a measurement of a velocity of light. The velocity of light is a universal constant for any inertial system.

The transformation which connects one inertial system to another inertial system is called Lorentz transformation. Therefore, the equation of motion of fields keeps its form unchanged under the Lorentz transformation.

1.2 Lorentz Transformation

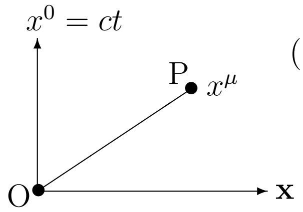
The principle of relativity is nicely formulated by introducing the 4-dimensional coordinate system x^μ $(\mu = 0, 1, 2, 3)$ which combines the time t and the space coordinate $\mathbf{x}$ as

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) = (ct, \mathbf{x}) \ , \tag{1.1}$$

where c is the velocity of light.

The coordinate x^μ specifies the point P in the 4-dimensional space. In this 4-dimensional space, 4-dimensional "distance" is introduced. The distance s from the origin O to the point P specified by the coordinate x^μ is defined as

$$\begin{aligned} s^2 &\equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = c^2t^2 - \mathbf{x}^2 \\ &= \sum_{\mu,\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu \ , \end{aligned} \tag{1.2}$$



where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.3}$$

is called metric tensor of the 4-dimensional space which measures the distance s between O and P from the difference of their coordinates. When $s^2 > 0$, the interval of O and P is called "time-like". When $s^2 < 0$, it is called "space-like". The special case $s^2 = 0$ is called "light-like".

Now, we write down the Lorentz transformation. Suppose one inertial system is described by the coordinate x^μ and another one by x'^μ. Since both x^μ and x'^μ are coordinate systems, their relation must be linear. Otherwise, the twice-length stick might not be measured as twice-length in another coordinate system. We assume the origin O of the 4-dimensional space is common for both systems. Therefore, the transformation should take a form

$$x^\mu \underset{\text{L.T.}}{\longrightarrow} x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu \ , \tag{1.4}$$

with the constant 4×4 matrix $\Lambda^\mu{}_\nu$. In addition, it must preserve the 4-dimensional distance (1.2):

$$s^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} x'^\mu x'^\nu = s'^2 \ . \tag{1.5}$$

This requirement comes from the motion of light. Suppose the light, passing through the origin O, arrives at the point P. For an observer in the inertial system with x^μ, the distance s between O and P should be $s = 0$, because the light moves along the straight line with the velocity c and therefore $|\mathbf{x}| = ct$. For another observer with x'^μ, this distance s' should also be $s' = 0$, because the velocity of light c is universal and he must observe $|\mathbf{x}'| = ct'$. The condition (1.5) assures $s = s' = 0$. Once this equality is realized for zero distance $s = 0$, the equality for nonzero distance $s = s' \neq 0$ results from the linearity of the transformation (1.4).

The 4-component quantity $A^\mu = (A^0, A^1, A^2, A^3) = (A^0, \mathbf{A})$ which transforms in the same way as x^μ is called Lorentz vector :

$$A^\mu \underset{\text{L.T.}}{\longrightarrow} A'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu A^\nu \ . \tag{1.6}$$

The square of A^μ defined below is invariant under Lorentz transformation as the distance s is invariant :

$$A^2 \equiv \sum_{\mu,\nu=0}^{3} g_{\mu\nu} A^\mu A^\nu = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} A'^\mu A'^\nu \equiv A'^2 \ . \tag{1.7}$$

The invariant quantity under Lorentz transformation is called Lorentz scalar. The scalar product of two Lorentz vectors A^μ and B^μ is defined by

$$A \cdot B \equiv A^0 B^0 - \mathbf{A} \cdot \mathbf{B} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} A^\mu B^\nu \ . \tag{1.8}$$

This is also Lorentz scalar because $(A+B)^2 = A^2 + B^2 + 2A \cdot B$ is Lorentz scalar.

It is convenient to introduce the subscript Lorentz vector A_μ by

$$A_\mu = (A_0, A_1, A_2, A_3) \equiv (A^0, -A^1, -A^2, -A^3) = (A^0, -\mathbf{A}) = g_{\mu\nu} A^\nu \ , \tag{1.9}$$

where, in the final expression, we have omitted the summation symbol $\Sigma_{\nu=0}^3$. Hereafter, when the same Greek indices appear in one term simultaneously as superscript and subscript indices, we assume that the summation over $0, 1, 2, 3$ is taken and we omit the symbol Σ. Thus, we write

$$A^\mu = g^{\mu\nu} A_\nu \ ; \qquad g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ , \tag{1.10}$$

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu = A_\mu B^\mu = g^{\mu\nu} A_\mu B_\nu \ . \tag{1.11}$$

Let us investigate the Lorentz transformation of the subscript vector A_μ. According to the definition, we have

$$A_\mu = g_{\mu\rho} A^\rho \underset{\text{L.T.}}{\longrightarrow} A'_\mu = g_{\mu\rho} A'^\rho = g_{\mu\rho} \Lambda^\rho{}_\sigma A^\sigma = g_{\mu\rho} \Lambda^\rho{}_\sigma g^{\sigma\nu} A_\nu \ . \tag{1.12}$$

Therefore, A_μ transforms as

$$A_\mu \underset{\text{L.T.}}{\longrightarrow} A'_\mu = \Lambda_\mu{}^\nu A_\nu \ ; \qquad \Lambda_\mu{}^\nu \equiv g_{\mu\rho} g^{\nu\sigma} \Lambda^\rho{}_\sigma \ . \tag{1.13}$$

Combining (1.6) and (1.13), we find that the scalar product of A^μ and B^μ transforms as

$$A_\mu B^\mu \underset{\text{L.T.}}{\longrightarrow} A'_\mu B'^\mu = \Lambda_\mu{}^\rho \Lambda^\mu{}_\sigma A_\rho B^\sigma \ . \tag{1.14}$$

This must be equal to

$$A_\mu B^\mu = \delta^\rho_\sigma A_\rho B^\sigma \ ; \qquad \delta^\rho_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{1.15}$$

Thus, we find the condition for the Lorentz transformation ;

$$\Lambda_\mu{}^\rho \Lambda^\mu{}_\sigma = \delta^\rho_\sigma \quad \text{and similarly} \quad \Lambda_\mu{}^\rho \Lambda^\nu{}_\rho = \delta^\nu_\mu \ . \tag{1.16}$$

Note that the inverse of the Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is expressed as

$$x^\nu = \Lambda_\mu{}^\nu x'^\mu \ . \tag{1.17}$$

Let us consider for a while a free motion of a particle with mass m. The dynamical quantities which characterize the motion of the particle are its energy E and momentum $\mathbf{p}$. They compose a Lorentz vector (4-momentum)

$$p^\mu = (E/c,\ \mathbf{p})\ . \tag{1.18}$$

The square of the 4-momentum p^μ is a Lorentz scalar and it is independent of the inertial system. From a dimensional consideration, it must be proportional to m^2c^2. In fact, it is

$$p^2 = E^2/c^2 - \mathbf{p}^2 = m^2c^2\ . \tag{1.19}$$

This is the so-called Einstein's relation. This relation should be regarded as a definition of m.

Suppose the particle is at rest ($\mathbf{p}$=0) in the inertial system $x^\mu_{(0)}$ and sitting on the origin O at the time $t_{(0)} = 0$. Its coordinate $x^\mu_{(0)}(\mathrm{O})$ and 4-momentum $p^\mu_{(0)}$ are $x^\mu_{(0)}(\mathrm{O}) = 0$ and $p^\mu_{(0)} = mc\ \delta^\mu_0$. After a time interval Δt, the particle "moves" to the point P specified by the coordinate $x^\mu_{(0)}(\mathrm{P}) = c\Delta t\delta^\mu_0$. In another inertial system, the particle is no longer at rest and all 4-vectors are transformed to

$$p^\mu = \Lambda^\mu{}_\nu p^\nu_{(0)} = mc\Lambda^\mu{}_0\ , \quad x^\mu(\mathrm{O}) = \Lambda^\mu{}_\nu x^\nu_{(0)}(\mathrm{O}) = 0\ , \quad x^\mu(\mathrm{P}) = \Lambda^\mu{}_\nu x^\nu_{(0)}(\mathrm{P}) = c\Delta t\Lambda^\mu{}_0\ . \tag{1.20}$$

Therefore, the velocity $\mathbf{v}$ of the particle is related to its energy E and momentum $\mathbf{p}$ by

$$v^i = \frac{x^i(\mathrm{P}) - x^i(\mathrm{O})}{(x^0(\mathrm{P}) - x^0(\mathrm{O}))/c} = c\cdot\frac{\Lambda^i{}_0}{\Lambda^0{}_0} = c\cdot\frac{p^i}{E/c}\ , \qquad \text{that is,}\quad \mathbf{v} = c^2\cdot\frac{\mathbf{p}}{E}\ . \tag{1.21}$$

Notice that the massless particle ($m = 0$) like photon moves with velocity $|\mathbf{v}| = c$.

1.3 Lorentz Transformation of Fields

Let $\phi(x)$ be the value of the field at the space-time point P described in one inertial system x^μ, and $\phi'(x')$ be its value at the same point P but in another inertial system x'^μ.

Scalar field does not change its value at any point P under the Lorentz transformation :

$$\phi(x) \underset{\text{L.T.}}{\longrightarrow} \phi'(x') = \phi(x) \qquad \text{space-time point P} \bullet^{x'^\mu}_{x^\mu} \tag{1.22}$$

Vector field changes values of its components according to the Lorentz transformation property :

$$\text{superscript 4-dim. vector}\quad A^\mu(x) \underset{\text{L.T.}}{\longrightarrow} A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)\ , \tag{1.23}$$

$$\text{subscript 4-dim. vector}\quad A_\mu(x) \underset{\text{L.T.}}{\longrightarrow} A'{}_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x)\ . \tag{1.24}$$

The derivative of scalar field by the coordinate x^μ transforms as

$$\frac{\partial\phi(x)}{\partial x^\mu} \underset{\text{L.T.}}{\longrightarrow} \frac{\partial\phi'(x')}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\phi(x)}{\partial x^\nu} = \Lambda_\mu{}^\nu\frac{\partial\phi(x)}{\partial x^\nu}\ , \tag{1.25}$$

where we have used (1.17). This expression invites us to introduce the notation

$$\partial_\mu\phi(x) = \frac{\partial\phi(x)}{\partial x^\mu}\ ; \qquad \partial_\mu = \left(\frac{\partial}{\partial x^0}\ ,\ \frac{\partial}{\partial x^1}\ ,\ \frac{\partial}{\partial x^2}\ ,\ \frac{\partial}{\partial x^3}\right) \equiv \left(\frac{\partial}{\partial x^0}\ ,\ \nabla\right)\ . \tag{1.26}$$

The differential ∂_μ transforms as a subscript Lorentz vector :

$$\partial_\mu \underset{\text{L.T.}}{\longrightarrow} \partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu\ ; \qquad \partial'_\mu = \frac{\partial}{\partial x'^\mu}\ . \tag{1.27}$$

The contracted second derivative $g^{\mu\nu}\partial_\mu\partial_\nu\phi(x)$ is a Lorentz scalar. The differential operator

$$\Box = g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{1.28}$$

is called d'Alembertian.

Chapter 2

Free Scalar Quantum Field

2.1 Classical Theory of Real Scalar Field

Let $\phi(\mathbf{x})$ be the scalar field which takes a real number as its value. Since $\phi(\mathbf{x})$ represents the continuously distributed dynamical degrees of freedom, the Lagrangian L of the system consists of the space integral of the Lagrangian density $\mathcal{L}(\mathbf{x})$:

$$L = \int d^3\mathbf{x}\ \mathcal{L}\left(\phi(\mathbf{x}), \dot{\phi}(\mathbf{x}), \nabla\phi(\mathbf{x})\right) \ . \tag{2.1}$$

Suppose the function $\phi(\mathbf{x}, t)$ is an arbitrary time trajectory of $\phi(\mathbf{x})$ which need not necessarily be the physical time development of $\phi(\mathbf{x})$. The action integral $I[\phi]$ of this trajectory is defined as a time integral of the Lagrangian L with $\phi(\mathbf{x})$ replaced by $\phi(\mathbf{x}, t)$;

$$I[\phi] = \int dt\ L = \int dt\, d^3\mathbf{x}\ \mathcal{L}\left(\phi(\mathbf{x}, t), \dot{\phi}(\mathbf{x}, t), \nabla\phi(\mathbf{x}, t)\right) \ . \tag{2.2}$$

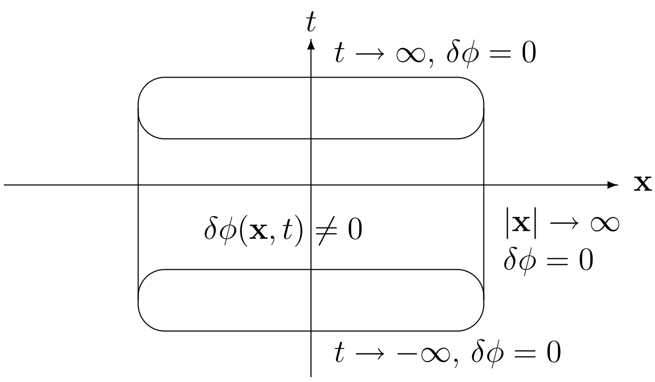
The basic principle of Nature recognized as the classical law of motion is summarized as the "Action Principle". The principle states, for the system of the field, that

The Classical Motion of the Field $\boldsymbol{\phi}(\mathbf{x})$ is Realized So That the Time Development $\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{t})$ Gives the Minimum Value of the Action Integral $\boldsymbol{I}[\boldsymbol{\phi}]$.

This means that the action integral $I[\phi]$ is stationary under any infinitesimal variation $\delta\phi$ of ϕ around physically realized trajectory $\phi(\mathbf{x}, t)$, subject to the constraint $\delta\phi(\mathbf{x}, t)\big|_{|t|, |\mathbf{x}| \to \infty,} = 0$, which is fixed by its boundary condition :

$$\phi(\mathbf{x}, t) \longrightarrow \phi(\mathbf{x}, t) + \delta\phi(\mathbf{x}, t) \ \ ; \ \begin{cases} \delta\phi(\mathbf{x}, t)\big|_{t\to\pm\infty} = 0 \\ \delta\phi(\mathbf{x}, t)\big|_{|\mathbf{x}|\to\infty} = 0 \end{cases} \tag{2.3}$$

$$\delta I \equiv I[\phi(\mathbf{x}, t) + \delta\phi(\mathbf{x}, t)] - I[\phi(\mathbf{x}, t)] = 0 \tag{2.4}$$



Notice that the principle is stated in terms of the "geometrical" terms independent of any inertial system. This is essential for the realization of the Lorentz invariance of the dynamics. The Lorentz invariance is realized only when the action integral $I[\phi]$ is Lorentz invariant. Then the minimum point of $I[\phi]$ is common for any observer in any inertial system.

Under the Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the 4-dimensional volume $cdtd^3\mathbf{x}$ transforms as

$$cdtd^3\mathbf{x} = \prod_{\mu=0}^{3} dx^\mu \longrightarrow \prod_{\mu=0}^{3} dx'^\mu = |\det \Lambda^\mu{}_\nu| \prod_{\mu=0}^{3} dx^\mu \ . \tag{2.5}$$

From (1.13) and (1.16), however, we find $|\det \Lambda^\mu{}_\nu| = 1$, and then the 4-dimensional volume is Lorentz invariant. Therefore, the Lagrangian density $\mathcal{L}$ must be Lorentz invariant. From now on, we simply call $\mathcal{L}$ Lagrangian instead of Lagrangian density. The simplest form of Lorentz invariant Lagrangian $\mathcal{L}$ which consists of ϕ, $\dot{\phi}$ and $\nabla\phi$ is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2}(\nabla\phi)^2 - V(\phi) \\ &= \frac{c^2}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \\ &\equiv \mathcal{L}(\phi, \partial_\mu\phi) \ ,\end{aligned} \tag{2.6}$$

where $V(\phi)$ is some polynomial of ϕ called potential.

Let us derive the consequence of the action principle for the general form of the Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi)$. Starting from (2.4) and utilizing partial integration, we obtain

$$\begin{aligned}0 = \delta I &= \int dtd^3\mathbf{x} \left(\frac{\partial\mathcal{L}}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} \underbrace{\delta(\partial_\mu\phi(x))}_{=\partial_\mu\delta\phi(x)} \right) \\ &= \int dtd^3\mathbf{x} \left(\frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} \right) \underbrace{\delta\phi(x)}_{\text{arbitrary}} + \underbrace{[\text{surface term}]}_{=0 \ \leftarrow \ \delta\phi|=0} \ .\end{aligned} \tag{2.7}$$

The surface term resulting from the partial integration vanishes owing to the boundary condition $\delta\phi|_{|t|,|\mathbf{x}|\to\infty} = 0$. Since the right-hand-side of (2.7) is required to vanish for any space-time dependent variation $\delta\phi(x)$, the quantity in the parenthesis must be zero at all space-time points x^μ :

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} - \frac{\partial\mathcal{L}}{\partial\phi(x)} = 0 \ . \tag{2.8}$$

This equation is called Euler-Lagrange equation, which serves as an equation of motion of the field $\phi(x)$. For the Lagrangian (2.6), we have

$$c^2 g^{\mu\nu}\partial_\mu\partial_\nu\phi(x) + \frac{dV}{d\phi} = 0 \quad \text{or equivalently} \quad \Box\phi + \frac{1}{c^2}\frac{dV}{d\phi} = 0 \ . \tag{2.9}$$

2.2 Canonical Quantization

For the quantization of the field $\phi(\mathbf{x})$, we start with giving a reminder of quantum mechanics.

Reminder of quantum mechanics

$$\text{generalized coordinates} \ : \ q_i \qquad (i = 1, \cdots, N) \ , \tag{2.10}$$

$$\text{generalized velocities} \;:\; \dot{q}_i \ , \tag{2.11}$$

$$\text{Lagrangian} \;:\; L(q_i, \dot{q}_i) \ , \tag{2.12}$$

$$\text{canonical momentum} \;:\; p_i = \frac{\partial L}{\partial \dot{q}_i} \ \Rightarrow \ \dot{q}_i = \dot{q}_i(q_i, p_i) \ , \tag{2.13}$$

$$\text{Hamiltonian} \;:\; H(p_i, q_i) = \sum_{i=1}^{N} p_i \dot{q}_i - L(q_i, \dot{q}_i) \ , \tag{2.14}$$

$$\text{canonical quantization} \;:\; q_i \Rightarrow \hat{q}_i \ , \ p_i \Rightarrow \hat{p}_i \ , \ H \Rightarrow \hat{H}(\hat{p}_i, \hat{q}_i) \ , \tag{2.15}$$

$\hat{f}$ represents the quantum operator corresponging to f.

$$\text{canonical commutator} \;:\; [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}, \ [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \ , \tag{2.16}$$

$$\text{Schrödinger equation} \;:\; i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \ . \tag{2.17}$$

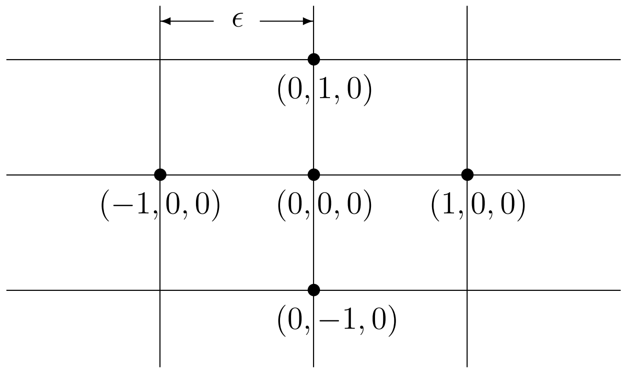
Since the field $\phi(\mathbf{x})$ is labeled by the continuous "index" $\mathbf{x}$, instead of the discrete index i, we need some prescription to apply the above quantization procedure. Among variety of possible prescriptions, we adopt here the so-called lattice discretization method. We discretize the three-dimensional space by the cubic lattice with infinitesimal lattice spacing ϵ. At each lattice point, we distribute the address $\boldsymbol{\alpha}$ which consists of a set of three integers $(\alpha_x, \alpha_y, \alpha_z)$. α_i's run from $-\infty$ to $+\infty$ by one unit successively in accordance with the ordering of the lattice points.

Discretization of the space by lattice

$\mathbf{x} \Rightarrow \boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$

Lattice point

$\alpha_i = -\infty, \cdots, -1, 0, 1, \cdots, +\infty$



Under this preparation, we represent the continuous degrees of freedom of the field $\phi(\mathbf{x})$ by the countable degrees of freedom $\phi_{\boldsymbol{\alpha}}$ located on each lattice point $\boldsymbol{\alpha}$. The correspondence between the continuous description and the discrete description is as follows ;

$$\begin{cases} \phi(\mathbf{x}) \Rightarrow \phi_{\boldsymbol{\alpha}} \\ \dot{\phi}(\mathbf{x}) \Rightarrow \dot{\phi}_{\boldsymbol{\alpha}} \\ \nabla_i \phi(\mathbf{x}) \Rightarrow \epsilon^{-1}(\phi_{\boldsymbol{\alpha}} - \phi_{\boldsymbol{\alpha}-\mathbf{n}_i}) \qquad i = x, y, z \quad \begin{cases} \mathbf{n}_x = (1,0,0) \\ \mathbf{n}_y = (0,1,0) \\ \mathbf{n}_z = (0,0,1) \end{cases} \\ \int d^3\mathbf{x} \Rightarrow \sum_{\boldsymbol{\alpha}} \epsilon^3 \end{cases} \tag{2.18}$$

Now, we are ready to apply the quantization procedure. The Lagrangian is expressed as

$$\begin{aligned} L = \int d^3\mathbf{x}\ \mathcal{L}(\mathbf{x}) &= \int d^3\mathbf{x} \left[\frac{1}{2}\dot{\phi}(\mathbf{x})^2 - \frac{c^2}{2} \left(\nabla\phi(\mathbf{x})\right)^2 - V(\phi(\mathbf{x})) \right] \\ &= \sum_{\boldsymbol{\alpha}} \epsilon^3 \left[\frac{1}{2}\dot{\phi}_{\boldsymbol{\alpha}}^2 - \frac{c^2}{2} \sum_{i=x,y,z} \left(\frac{\phi_{\boldsymbol{\alpha}} - \phi_{\boldsymbol{\alpha}-\mathbf{n}_i}}{\epsilon} \right)^2 - V(\phi_{\boldsymbol{\alpha}}) \right] \ . \end{aligned} \tag{2.19}$$

The canonical momentum conjugate to $\phi_{\boldsymbol{\alpha}}$ is

$$p_{\boldsymbol{\alpha}} \equiv \frac{\partial L}{\partial \dot{\phi}_{\boldsymbol{\alpha}}} = \epsilon^3 \dot{\phi}_{\boldsymbol{\alpha}} \equiv \epsilon^3 \pi_{\boldsymbol{\alpha}} \ , \tag{2.20}$$

where we have introduced $\pi_{\boldsymbol{\alpha}}$ as a rescaled quantity of $p_{\boldsymbol{\alpha}}$ so that it survives in the limit $\epsilon \to 0$. The Hamiltonian is given by

$$\begin{aligned} H &= \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \dot{\phi}_{\boldsymbol{\alpha}} - L \\ &= \sum_{\boldsymbol{\alpha}} \epsilon^3 \left[\frac{1}{2}\pi_{\boldsymbol{\alpha}}^2 + \frac{c^2}{2} \sum_{i=x,y,z} \left(\frac{\phi_{\boldsymbol{\alpha}} - \phi_{\boldsymbol{\alpha}-\mathbf{n}_i}}{\epsilon} \right)^2 + V(\phi_{\boldsymbol{\alpha}}) \right] \\ &= \int d^3\mathbf{x} \left[\frac{1}{2}\pi(\mathbf{x})^2 + \frac{c^2}{2} \left(\nabla\phi(\mathbf{x})\right)^2 + V(\phi(\mathbf{x})) \right] \\ &\equiv \int d^3\mathbf{x} \ \mathcal{H}(\mathbf{x}) \ , \end{aligned} \tag{2.21}$$

where we have returned to the continuous description, with an additional correspondence

$$\pi_{\boldsymbol{\alpha}} \Rightarrow \pi(\mathbf{x}) \ . \tag{2.22}$$

The field $\pi(\mathbf{x})$ is called canonical momentum density.

Quantization : The quantization to $\hat{\phi}_{\boldsymbol{\alpha}}$ and $\hat{p}_{\boldsymbol{\alpha}}$ is straightforward. The canonical commutators are

$$[\hat{\phi}_{\boldsymbol{\alpha}}, \hat{p}_{\boldsymbol{\beta}}] = i\hbar\delta^3_{\boldsymbol{\alpha\beta}} \ , \quad [\hat{\phi}_{\boldsymbol{\alpha}}, \hat{\phi}_{\boldsymbol{\beta}}] = [\hat{p}_{\boldsymbol{\alpha}}, \hat{p}_{\boldsymbol{\beta}}] = 0 \ ; \qquad \delta^3_{\boldsymbol{\alpha\beta}} \equiv \delta_{\alpha_x\beta_x}\delta_{\alpha_y\beta_y}\delta_{\alpha_z\beta_z} \ . \tag{2.23}$$

From the relation $\hat{p}_{\boldsymbol{\alpha}} = \epsilon^3 \hat{\pi}_{\boldsymbol{\alpha}}$, we have

$$[\hat{\phi}_{\boldsymbol{\alpha}}, \hat{\pi}_{\boldsymbol{\beta}}] = i\hbar \frac{1}{\epsilon^3} \delta^3_{\boldsymbol{\alpha\beta}} \ . \tag{2.24}$$

The limit $\epsilon \to 0$ of the right-hand-side is expressed in terms of the Dirac's delta function $\delta^3(\mathbf{x})$;

$$\frac{1}{\epsilon^3}\delta^3_{\boldsymbol{\alpha\beta}} \underset{\epsilon\to 0}{\longrightarrow} \delta^3(\mathbf{x}-\mathbf{y}) \ . \tag{2.25}$$

Thus, we obtain the continuous expression for the commutation relations :

$$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\hbar\delta^3(\mathbf{x}-\mathbf{y}) \ , \tag{2.26}$$

$$[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0 \ . \tag{2.27}$$

Hamiltonian Operator : The Hamiltonian operator is obtained by simply replacing $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ in (2.21) with the corresponding quantum operators $\hat{\phi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$;

$$\hat{H} = \int d^3\mathbf{x} \ \hat{\mathcal{H}}(\mathbf{x}) = \int d^3\mathbf{x} \left[\frac{1}{2}\hat{\pi}(\mathbf{x})^2 + \frac{c^2}{2} \left(\nabla\hat{\phi}(\mathbf{x})\right)^2 + V(\hat{\phi}(\mathbf{x})) \right] \tag{2.28}$$

Remember that we are treating a classically real scalar field, that is, ϕ and π are real quantities. Therefore, the quantum operators $\hat{\phi}$ and $\hat{\pi}$ are hermitian operators ;

$$\hat{\phi}^\dagger = \hat{\phi} \ , \quad \hat{\pi}^\dagger = \hat{\pi} \ . \tag{2.29}$$

The Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{2.30}$$

determines the time development of the quantum state $|\psi(t)\rangle$ of the dynamical system which consists of the field $\hat{\phi}$ and $\hat{\pi}$.

Momentum Operator : The operator $\hat{\mathbf{P}}$, which realizes the commutation relations

$$[\hat{\mathbf{P}}, \hat{\phi}(\mathbf{x})] = i\hbar\nabla\hat{\phi}(\mathbf{x}) \ , \quad [\hat{\mathbf{P}}, \hat{\pi}(\mathbf{x})] = i\hbar\nabla\hat{\pi}(\mathbf{x}) \ , \tag{2.31}$$

is called momentum operator. It is given by

$$\hat{\mathbf{P}} = -\int d^3\mathbf{x} \ \hat{\pi}(\mathbf{x})\nabla\hat{\phi}(\mathbf{x}) \ . \tag{2.32}$$

We can easily confirm that $\hat{\mathbf{P}}$ given by (2.32) reproduces the commutators (2.31). For example,

$$\begin{aligned} [\hat{\mathbf{P}}, \hat{\pi}(\mathbf{x})] &= -\int d^3\mathbf{y} \ [\hat{\pi}(\mathbf{y})\nabla_{(y)}\hat{\phi}(\mathbf{y}), \hat{\pi}(\mathbf{x})] = -\int d^3\mathbf{y} \ \hat{\pi}(\mathbf{y})\nabla_{(y)}[\hat{\phi}(\mathbf{y}), \hat{\pi}(\mathbf{x})] \\ &= -\int d^3\mathbf{y} \ \hat{\pi}(\mathbf{y})\nabla_{(y)} i\hbar\delta^3(\mathbf{y}-\mathbf{x}) = i\hbar\nabla\hat{\pi}(\mathbf{x}) \ ; \qquad \nabla_{(y)} \equiv \frac{\partial}{\partial\mathbf{y}} \ , \end{aligned} \tag{2.33}$$

where, in the second line, we have used a partial integration.

The remarkable property of the momentum operator $\hat{\mathbf{P}}$ is that it realizes the commutation relation

$$[\hat{\mathbf{P}}, \hat{O}(\mathbf{x})] = i\hbar\nabla\hat{O}(\mathbf{x}) \tag{2.34}$$

for any operator $\hat{O}(\mathbf{x})$ which is an arbitrary product of $\hat{\phi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$. This is because, for $\hat{O} = \hat{O}_1\hat{O}_2$, we have

$$[\hat{\mathbf{P}}, \hat{O}] = [\hat{\mathbf{P}}, \hat{O}_1]\hat{O}_2 + \hat{O}_1[\hat{\mathbf{P}}, \hat{O}_2] = (i\hbar\nabla\hat{O}_1)\hat{O}_2 + \hat{O}_1(i\hbar\nabla\hat{O}_2) = i\hbar\nabla(\hat{O}_1\hat{O}_2) \ . \tag{2.35}$$

According to the formula of similarity transformation

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \cdots \ , \tag{2.36}$$

we have

$$e^{-i\frac{1}{\hbar}\mathbf{a}\cdot\hat{\mathbf{P}}}\hat{O}(\mathbf{x})e^{i\frac{1}{\hbar}\mathbf{a}\cdot\hat{\mathbf{P}}} = \hat{O}(\mathbf{x}) + \mathbf{a}\cdot\nabla\hat{O}(\mathbf{x}) + \frac{1}{2!}(\mathbf{a}\cdot\nabla)^2\hat{O}(\mathbf{x}) + \cdots = \hat{O}(\mathbf{x}+\mathbf{a}) \tag{2.37}$$

for any constant vector $\mathbf{a}$. Since the Hamiltonian $\hat{H}$ is a spacial integral of the Hamiltonian density $\hat{\mathcal{H}}(\mathbf{x})$, we find

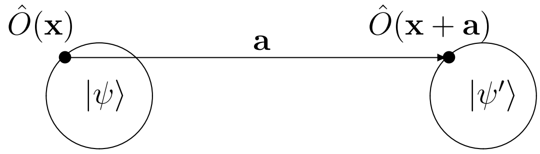
$$e^{-i\frac{1}{\hbar}\mathbf{a}\cdot\hat{\mathbf{P}}}\hat{H}e^{i\frac{1}{\hbar}\mathbf{a}\cdot\hat{\mathbf{P}}} = \int d^3\mathbf{x} \ \hat{\mathcal{H}}(\mathbf{x}+\mathbf{a}) = \int d^3\mathbf{x} \ \hat{\mathcal{H}}(\mathbf{x}) = \hat{H} \ . \tag{2.38}$$

Owing to the arbitrariness of the vector $\mathbf{a}$, this leads us to the important observation

$$[\hat{\mathbf{P}}, \hat{H}] = 0 \ . \tag{2.39}$$

For an arbitrary state $|\psi\rangle$, let us define $|\psi'\rangle = e^{-i\frac{1}{\hbar}\mathbf{a}\cdot\hat{\mathbf{P}}}|\psi\rangle$. Then we have

$$\langle\psi|\hat{O}(\mathbf{x})|\psi\rangle = \langle\psi'|\hat{O}(\mathbf{x}+\mathbf{a})|\psi'\rangle \ . \tag{2.40}$$



This means that the state $|\psi'\rangle$ is a state obtained by translating the state $|\psi\rangle$ by a distance $\mathbf{a}$. Thus, $\hat{\mathbf{P}}$ is also called a space translation operator.

Heisenberg Picture : The integral of the Schrödinger equation is given by

$$|\psi(t)\rangle = e^{-i\frac{1}{\hbar}\hat{H}t}|\psi(0)\rangle \ . \tag{2.41}$$

Suppose the states $|\psi(t)\rangle$ and $|\psi'(t)\rangle$ are the solutions of the Schrödinger equation specified by the initial states $|\psi(0)\rangle$ and $|\psi'(0)\rangle$, respectively, at $t = 0$. The matrix element of any operator $\hat{O}$ between $|\psi(t)\rangle$ and $|\psi'(t)\rangle$ is expressed as

$$\langle\psi'(t)|\hat{O}|\psi(t)\rangle = \langle\psi'(0)|\hat{O}(t)|\psi(0)\rangle \ . \tag{2.42}$$

This defines the operator $\hat{O}(t)$ called Heisenberg operator :

$$\hat{O}(t) = e^{i\frac{1}{\hbar}\hat{H}t}\hat{O}e^{-i\frac{1}{\hbar}\hat{H}t} \ . \tag{2.43}$$

The left-hand-side of (2.42) is viewing the time development of the matrix element through the time evolution of the state vectors. This point of view is called Schrödinger picture. The right-hand-side on the other hand is viewing the same matrix element through the "motion" of the operator $\hat{O}(t)$, with the state vectors fixed at $t = 0$. This point of view is called Heisenberg picture.

The Heisenberg operator of a product of two operators $\hat{O}_1$ and $\hat{O}_2$ is a product of the corresponding Heisenberg operators ;

$$\widehat{O_1 O_2}(t) = \hat{O}_1(t)\hat{O}_2(t) \ . \tag{2.44}$$

Equal Time Commutation Relation : The operators $\hat{\phi}(\mathbf{x}, t)$ and $\hat{\pi}(\mathbf{x}, t)$ in the Heisenberg picture satisfy the equal time commutation relations

$$[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)] = i\hbar\delta^3(\mathbf{x}-\mathbf{y}) \ , \tag{2.45}$$

$$[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{y}, t)] = [\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)] = 0 \ . \tag{2.46}$$

Since $\hat{H}(t) = \hat{H}$, we have

$$\hat{H}(t) = \int d^3\mathbf{x} \left[\frac{1}{2}\hat{\pi}(\mathbf{x}, t)^2 + \frac{c^2}{2}\left(\nabla\hat{\phi}(\mathbf{x}, t)\right)^2 + V(\hat{\phi}(\mathbf{x}, t))\right] = \hat{H} \ . \tag{2.47}$$

The commutation relation (2.39) also states $\hat{\mathbf{P}}(t) = \hat{\mathbf{P}}$, and we have

$$\hat{\mathbf{P}}(t) = -\int d^3\mathbf{x}\ \hat{\pi}(\mathbf{x}, t)\nabla\hat{\phi}(\mathbf{x}, t) = \hat{\mathbf{P}} \ . \tag{2.48}$$

These two equations represent the conservation of energy and momentum.

Heisenberg's Equation of Motion : Taking a time derivative of the Heisenberg operator (2.43), we obtain the equation of motion for the Heisenberg operator $\hat{O}(t)$;

$$\dot{\hat{O}}(t) = -\frac{i}{\hbar}[\hat{O}(t), \hat{H}(t)] \ . \tag{2.49}$$

For the local operator $\hat{O}(\mathbf{x},t)$, this equation and the equation (2.34) represented in terms of the Heisenberg operators are combined to the Lorentz covariant equation ;

$$i\hbar\partial^\mu\hat{O}(\mathbf{x},t) = [\hat{O}(\mathbf{x},t), \hat{P}^\mu] \ , \quad \hat{P}^\mu = (\hat{H}/c, \ \hat{\mathbf{P}}) \ , \quad \partial^\mu = \big(\partial/\partial x^0, \ -\nabla\big) \tag{2.50}$$

Note the derivative by the coordinate is originally defined as a subscript vector $\partial_\mu = (\partial/\partial x^0, \nabla)$.

Let us derive the equations of motion for $\hat{\phi}(\mathbf{x},t)$ and $\hat{\pi}(\mathbf{x},t)$. Noticing $[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{y},t)] = [\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = 0$, we have

$$\dot{\hat{\phi}}(\mathbf{x},t) = -\frac{i}{\hbar}\left[\hat{\phi}(\mathbf{x},t), \int d^3\mathbf{y}\frac{1}{2}\hat{\pi}(\mathbf{y},t)^2\right] \ , \tag{2.51}$$

$$\dot{\hat{\pi}}(\mathbf{x},t) = -\frac{i}{\hbar}\left[\hat{\pi}(\mathbf{x},t), \int d^3\mathbf{y}\left(\frac{c^2}{2}\nabla_{(y)}\hat{\phi}(\mathbf{y},t)\cdot\nabla_{(y)}\hat{\phi}(\mathbf{y},t) + V(\hat{\phi}(\mathbf{y},t))\right)\right] \ . \tag{2.52}$$

The identity $[\hat{X}, \hat{Y}\hat{Z}] = [\hat{X}, \hat{Y}]\hat{Z} + \hat{Y}[\hat{X}, \hat{Z}]$ gives us the commutators

$$\left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)^2\right] = [\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)]\hat{\pi}(\mathbf{y},t) + \hat{\pi}(\mathbf{y},t)[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = 2i\hbar\delta^3(\mathbf{x}-\mathbf{y})\hat{\pi}(\mathbf{y},t), \tag{2.53}$$

$$\left[\hat{\pi}(\mathbf{x},t), \nabla_{(y)}\hat{\phi}(\mathbf{y},t)\cdot\nabla_{(y)}\hat{\phi}(\mathbf{y},t)\right] = -2i\hbar\nabla_{(y)}\delta^3(\mathbf{y}-\mathbf{x})\cdot\nabla\hat{\phi}(\mathbf{y},t) \ . \tag{2.54}$$

For the potential $V(\hat{\phi}(\mathbf{y},t)) = \sum_N v_N\hat{\phi}(\mathbf{y},t)^N$, the identity $[\hat{X}, \hat{Y}^N] = N[\hat{X}, \hat{Y}]\hat{Y}^{N-1}$, which holds when $[\hat{X}, \hat{Y}]$ is a c-number, gives

$$\begin{aligned}[\hat{\pi}(\mathbf{x},t), V(\hat{\phi}(\mathbf{y},t))] &= \sum_N v_N[\hat{\pi}(\mathbf{x},t), \hat{\phi}(\mathbf{y},t)^N] = \sum_N v_N N(-i\hbar)\delta^3(\mathbf{y}-\mathbf{x})\hat{\phi}(\mathbf{y},t)^{N-1} \\ &= -i\hbar\delta^3(\mathbf{y}-\mathbf{x})\frac{dV(\hat{\phi}(\mathbf{y},t))}{d\phi} \ . \end{aligned} \tag{2.55}$$

Gathering all together, we obtain the Heisenberg's equations of motion for $\hat{\phi}(\mathbf{x},t)$ and $\hat{\pi}(\mathbf{x},t)$;

$$\dot{\hat{\phi}}(\mathbf{x},t) = \hat{\pi}(\mathbf{x},t) \ , \qquad \dot{\hat{\pi}}(\mathbf{x},t) = c^2\nabla^2\hat{\phi}(\mathbf{x},t) - \frac{dV(\hat{\phi}(\mathbf{x},t))}{d\phi} \ . \tag{2.56}$$

Combining these two equations, we arrive at the equation

$$\ddot{\hat{\phi}}(\mathbf{x},t) = \dot{\hat{\pi}}(\mathbf{x},t) = c^2\nabla^2\hat{\phi}(\mathbf{x},t) - \frac{dV(\hat{\phi}(\mathbf{x},t))}{d\phi} \ . \tag{2.57}$$

This reproduces the original Euler-Lagrange equation $\Box\hat{\phi} + \frac{1}{c^2}\frac{d\hat{V}}{d\phi} = 0$ in terms of $\hat{\phi}(\mathbf{x},t)$.

2.3 Free Scalar Field

From now on, we use the Natural unit system $c = 1$, $\hbar = 1$.

Klein-Gordon Equation : Field ϕ whose equation of motion is linear in ϕ is called free field. Thus the potential V is quadratic

$$V = \frac{1}{2}m^2\phi^2 \ . \tag{2.58}$$

The equation of motion

$$(\Box + m^2)\hat{\phi}(\mathbf{x},t) = 0 \tag{2.59}$$

is called Klein-Gordon equation. m expresses the mass of the particle the field $\hat{\phi}$ describes.

Fourier transform : Let us express the field operator $\hat{\phi}(\mathbf{x}, t)$ in the Fourier integral in terms of the plane-wave function $\exp(i\mathbf{k}\cdot\mathbf{x})$;

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\phi}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}} . \tag{2.60}$$

The operator property of $\hat{\phi}(\mathbf{x}, t)$ is inherited by $\hat{\phi}_{\mathbf{k}}(t)$. The hermiticity of $\hat{\phi}$ is translated into

$$\hat{\phi}^\dagger_{\mathbf{k}}(t) = \hat{\phi}_{-\mathbf{k}}(t) . \tag{2.61}$$

Substituting $\hat{\phi}(\mathbf{x}, t)$ in the Klein-Gordon equation (2.59) by the expression (2.60), we have

$$\begin{aligned} 0 &= (\Box + m^2)\hat{\phi}(\mathbf{x}, t) \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(\ddot{\hat{\phi}}_{\mathbf{k}}(t) + (\mathbf{k}^2 + m^2)\hat{\phi}_{\mathbf{k}}(t)\right) e^{i\mathbf{k}\cdot\mathbf{x}} . \end{aligned} \tag{2.62}$$

Since the plane-wave functions form a complete set of orthonormal functions, this requires

$$\ddot{\hat{\phi}}_{\mathbf{k}}(t) + (\mathbf{k}^2 + m^2)\hat{\phi}_{\mathbf{k}}(t) = 0 . \tag{2.63}$$

This is nothing but an equation of motion of the harmonic oscilator with angular frequency

$$\omega_k = \sqrt{\mathbf{k}^2 + m^2} . \tag{2.64}$$

The solution which satisfies the hermiticity condition (2.61) is

$$\hat{\phi}_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega_k}}(\hat{a}_{\mathbf{k}} e^{-i\omega_k t} + \hat{a}^\dagger_{-\mathbf{k}} e^{i\omega_k t}) , \tag{2.65}$$

where we have introduced a normalization constant $1/\sqrt{2\omega_k}$ for the later convenience. Thus, we obtain

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right) , \tag{2.66}$$

$$\hat{\pi}(\mathbf{x}, t) = \dot{\hat{\phi}}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left(-i\hat{a}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + i\hat{a}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right) , \tag{2.67}$$

where, in the terms originally containing $\hat{a}^\dagger_{-\mathbf{k}}$, we have replaced the integration variable $\mathbf{k}$ by $-\mathbf{k}$.

Commutator of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$: The commutation relations (2.45) and (2.46) of $\hat{\phi}$ and $\hat{\pi}$ determine the commutation relations of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$. The result is

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}') , \tag{2.68}$$

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0 . \tag{2.69}$$

We can easily confirm that this really reproduces (2.45) and (2.46). For example,

$$\begin{aligned} [\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)] &= \int \frac{d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^3} \frac{1}{2} \sqrt{\frac{\omega_{k'}}{\omega_k}} \Big\{ i[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] e^{-i((\omega_k - \omega_{k'})t - \mathbf{k}\cdot\mathbf{x} + \mathbf{k}'\cdot\mathbf{y})} \\ &\qquad - i[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] e^{i((\omega_k - \omega_{k'})t - \mathbf{k}\cdot\mathbf{x} + \mathbf{k}'\cdot\mathbf{y})} \Big\} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2} \left(i e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} + i e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\right) \\ &= i \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= i\delta^3(\mathbf{x} - \mathbf{y}) . \end{aligned} \tag{2.70}$$

Let us express the Hamiltonian

$$\hat{H}(t) = \int d^3\mathbf{x} \left(\frac{1}{2}\hat{\pi}(\mathbf{x},t)^2 + \frac{1}{2}\left(\nabla\hat{\phi}(\mathbf{x},t)\right)^2 + \frac{1}{2}m^2\hat{\phi}(\mathbf{x},t)^2 \right) \tag{2.71}$$

in terms of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$. By substituting the expressions for $\hat{\phi}(\mathbf{x},t)$ and $\hat{\pi}(\mathbf{x},t)$, we have

$$\begin{aligned}
\hat{H}(t) = &\int d^3\mathbf{x}\, \frac{1}{2} \int \frac{d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^3} \frac{1}{2\sqrt{\omega_k \omega_{k'}}} \\
&\times \Big[\omega_k \omega_{k'} \left(-i\hat{a}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + i\hat{a}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} \right) \left(-i\hat{a}_{\mathbf{k}'} e^{-i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{x})} + i\hat{a}^\dagger_{\mathbf{k}'} e^{i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{x})} \right) \\
&\quad + \mathbf{k}\cdot\mathbf{k}' \left(i\hat{a}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} - i\hat{a}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} \right) \left(i\hat{a}_{\mathbf{k}'} e^{-i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{x})} - i\hat{a}^\dagger_{\mathbf{k}'} e^{i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{x})} \right) \\
&\quad + m^2 \left(\hat{a}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} \right) \left(\hat{a}_{\mathbf{k}'} e^{-i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{x})} + \hat{a}^\dagger_{\mathbf{k}'} e^{i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{x})} \right) \Big] \\
= &\frac{1}{2} \int d^3\mathbf{k} d^3\mathbf{k}' \frac{1}{2\sqrt{\omega_k \omega_{k'}}} \int \frac{d^3\mathbf{x}}{(2\pi)^3} \\
&\times \Big[(\omega_k \omega_{k'} + \mathbf{k}\cdot\mathbf{k}' + m^2) \left(\hat{a}_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}'} e^{-i(\omega_k - \omega_{k'})t} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}'} e^{i(\omega_k - \omega_{k'})t} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \right) \\
&\quad + (-\omega_k \omega_{k'} - \mathbf{k}\cdot\mathbf{k}' + m^2) \left(\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} e^{-i(\omega_k + \omega_{k'})t} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}'} e^{i(\omega_k + \omega_{k'})t} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \right) \Big] \\
= &\frac{1}{2} \int d^3\mathbf{k}\, \frac{1}{2\omega_k} \Big[(\omega_k^2 + \mathbf{k}^2 + m^2) \left(\hat{a}_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \right) \\
&\quad + (-\omega_k^2 + \mathbf{k}^2 + m^2) \left(\hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} e^{-2i\omega_k t} + \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} e^{2i\omega_k t} \right) \Big] ,
\end{aligned} \tag{2.72}$$

where we have used the formula

$$\int \frac{d^3\mathbf{x}}{(2\pi)^3}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = \delta^3(\mathbf{k}-\mathbf{k}') \,. \tag{2.73}$$

Since $\omega_k^2 = \mathbf{k}^2 + m^2$, the second term in the bracket vanishes, and we find

$$\begin{aligned}
\hat{H}(t) &= \int d^3\mathbf{k} \frac{\omega_k}{2} (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}}) \\
&= \int d^3\mathbf{k}\; \omega_k \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + [\text{const.}] \,.
\end{aligned} \tag{2.74}$$

The term [const.] in the final expression appears due to the interchange of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$. Obviously, it is a collection of the zero-point energies of the harmonic oscillators $\hat{a}_{\mathbf{k}}$. It is expressed as

$$[\text{const.}] = \int d^3\mathbf{k} \frac{\omega_k}{2} \delta^3(\mathbf{k}-\mathbf{k}) = \int d^3\mathbf{k} \frac{\omega_k}{2} \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{i(\mathbf{k}-\mathbf{k})\cdot\mathbf{x}} = V \int d^3\mathbf{k} \frac{\omega_k}{16\pi^3} \,, \tag{2.75}$$

where V is the total volume of the three-dimensional space. Thus, the space posseses the energy $\epsilon_{\text{vac}} = \int d^3\mathbf{k}\; \omega_k/(16\pi^3)$ per unit volume. Since ω_k is given by $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$, the energy density ϵ_{vac} is quarticaly divergent. This is the first divergence we encounter in the dynamical system of quantum field. As a matter of fact, [const.] plays no significant role in quantum field theory, because it is merely a constant, though divergent. We can always shift the origin of energy so that [const.]=0 keeping intact any physical content of the system. So, hereafter we omit it.

Next, let us turn to the momentum operator $\hat{\mathbf{P}}$. We leave the details of the calculation for the exercise. The result is

$$\begin{aligned}\hat{\mathbf{P}}(t) &= -\int d^3\mathbf{x}\hat{\pi}(\mathbf{x},t)\nabla\hat{\phi}(\mathbf{x},t) \\ &= \int d^3\mathbf{k}\frac{\mathbf{k}}{2}(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}) \\ &= \int d^3\mathbf{k}\ \mathbf{k}\ \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\ . \end{aligned} \tag{2.76}$$

At this time, we do not have [const.] term because the integrand contains an odd quantity $\mathbf{k}$. Notice that t dependence disapeared in the final expressions of $\hat{H}(t)$ and $\hat{\mathbf{P}}(t)$ as it should be.

Creation and Annihilation Operator : The Hamiltonian $\hat{H}$ and momentum $\hat{\mathbf{P}}$ contain the product $\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}$. The commutation relations of this operator with $\hat{a}_{\mathbf{k}'}$ and $\hat{a}^\dagger_{\mathbf{k}'}$ are

$$[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}]\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = -\delta^3(\mathbf{k}-\mathbf{k}')\hat{a}_{\mathbf{k}}\ , \tag{2.77}$$
$$[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}]\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k}-\mathbf{k}')\hat{a}^\dagger_{\mathbf{k}}\ . \tag{2.78}$$

From this result, we find the following commutation relations :

$$[\hat{H}, \hat{a}_{\mathbf{k}}] = -\omega_k\hat{a}_{\mathbf{k}}\ , \quad [\hat{H}, \hat{a}^\dagger_{\mathbf{k}}] = \omega_k\hat{a}^\dagger_{\mathbf{k}}\ , \tag{2.79}$$
$$[\hat{\mathbf{P}}, \hat{a}_{\mathbf{k}}] = -\mathbf{k}\hat{a}_{\mathbf{k}}\ , \quad [\hat{\mathbf{P}}, \hat{a}^\dagger_{\mathbf{k}}] = \mathbf{k}\hat{a}^\dagger_{\mathbf{k}}\ . \tag{2.80}$$

These commutation relations play the fundamental role as a bridge between fields and particles. Suppose a state $|E,\mathbf{P}\rangle$ is an eigenstate of $\hat{H}$ and $\hat{\mathbf{P}}$:

$$\hat{H}|E,\mathbf{P}\rangle = E|E,\mathbf{P}\rangle\ , \qquad \hat{\mathbf{P}}|E,\mathbf{P}\rangle = \mathbf{P}|E,\mathbf{P}\rangle\ . \tag{2.81}$$

Then we have

$$\hat{H}\hat{a}_{\mathbf{k}}|E,\mathbf{P}\rangle = \left([\hat{H},\hat{a}_{\mathbf{k}}] + \hat{a}_{\mathbf{k}}\hat{H}\right)|E,\mathbf{P}\rangle = (E-\omega_k)\hat{a}_{\mathbf{k}}|E,\mathbf{P}\rangle\ , \tag{2.82}$$
$$\hat{H}\hat{a}^\dagger_{\mathbf{k}}|E,\mathbf{P}\rangle = \left([\hat{H},\hat{a}^\dagger_{\mathbf{k}}] + \hat{a}_{\mathbf{k}}\hat{H}\right)|E,\mathbf{P}\rangle = (E+\omega_k)\hat{a}^\dagger_{\mathbf{k}}|E,\mathbf{P}\rangle\ . \tag{2.83}$$

Similarly,

$$\hat{\mathbf{P}}\hat{a}_{\mathbf{k}}|E,\mathbf{P}\rangle = (\mathbf{P}-\mathbf{k})\hat{a}_{\mathbf{k}}|E,\mathbf{P}\rangle\ , \qquad \hat{\mathbf{P}}\hat{a}^\dagger_{\mathbf{k}}|E,\mathbf{P}\rangle = (\mathbf{P}+\mathbf{k})\hat{a}^\dagger_{\mathbf{k}}|E,\mathbf{P}\rangle\ . \tag{2.84}$$

That is, $\hat{a}_{\mathbf{k}}$, when operated to a state, decreases and $\hat{a}^\dagger_{\mathbf{k}}$ increases the energy and momentum of the state by ω_k and $\mathbf{k}$, respectively. The energy ω_k and the momentum $\mathbf{k}$ satisfy the Einstein's relation

$$\omega_k^2 - \mathbf{k}^2 = m^2 \tag{2.85}$$

of a particle with mass m. This shows that $\hat{a}_{\mathbf{k}}$ annihilates and $\hat{a}^\dagger_{\mathbf{k}}$ creates the particle with energy ω_k and momentum $\mathbf{k}$. Thus, $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ are called creation and annihilation operator, respectively.

Space of Quantum States (Fock Space) : The state which plays a basis of the system of quantum field is the normalized ground state $|0\rangle$ of the Hamiltonian $\hat{H}$;

$$\hat{H}|0\rangle = E_0|0\rangle\ , \qquad \langle 0|0\rangle = 1\ . \tag{2.86}$$

Evidently, it should be a state where no particle exists. That is, no particle can be annihilated by any of the annihilation operator $\hat{a}_{\mathbf{k}}$;

$$\hat{a}_{\mathbf{k}}|0\rangle = 0\ . \tag{2.87}$$

The state $|0\rangle$ is called the vacuum of the quantum system. From (2.74) and (2.76), we find, keeping in mind of our convention [const.]=0,

$$\hat{H}|0\rangle = 0 \ , \qquad \hat{\mathbf{P}}|0\rangle = 0 \ . \tag{2.88}$$

Any state other than the vacuum $|0\rangle$ is obtained by operating the creation operators $\hat{a}^\dagger_{\mathbf{k}}$ on $|0\rangle$. The one particle state is

$$|\mathbf{k}\rangle = \hat{a}^\dagger_{\mathbf{k}}|0\rangle \ , \tag{2.89}$$

$$\hat{H}|\mathbf{k}\rangle = \omega_k|\mathbf{k}\rangle \ , \ \ \hat{\mathbf{P}}|\mathbf{k}\rangle = \mathbf{k}|\mathbf{k}\rangle \ , \qquad \omega_k = \sqrt{\mathbf{k}^2 + m^2} \ . \tag{2.90}$$

Owing to the commutator (2.68) and the vacuum condition (2.87), $|\mathbf{k}\rangle$ satisfies the orthonormality

$$\langle\mathbf{k}|\mathbf{k}'\rangle = \langle 0|\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}'}|0\rangle = \langle 0|[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] + \hat{a}^\dagger_{\mathbf{k}'}\hat{a}_{\mathbf{k}}|0\rangle = \delta^3(\mathbf{k} - \mathbf{k}') \ . \tag{2.91}$$

The two particle state is

$$|\mathbf{k}_1, \mathbf{k}_2\rangle = \hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}|0\rangle \ , \tag{2.92}$$

$$\hat{H}|\mathbf{k}_1, \mathbf{k}_2\rangle = (\omega_{k_1} + \omega_{k_2})|\mathbf{k}_1, \mathbf{k}_2\rangle \ , \ \ \hat{\mathbf{P}}|\mathbf{k}_1, \mathbf{k}_2\rangle = (\mathbf{k}_1 + \mathbf{k}_2)|\mathbf{k}_1, \mathbf{k}_2\rangle \ , \ \ \omega_{k_i} = \sqrt{\mathbf{k}_i^2 + m^2} \ . \tag{2.93}$$

Due to the commutability (2.69) of $\hat{a}^\dagger_{\mathbf{k}_1}$ and $\hat{a}^\dagger_{\mathbf{k}_2}$, the state $|\mathbf{k}_1, \mathbf{k}_2\rangle$ is symmetric under the interchange of $\mathbf{k}_1$ and $\mathbf{k}_2$;

$$|\mathbf{k}_1, \mathbf{k}_2\rangle = \hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}|0\rangle = \hat{a}^\dagger_{\mathbf{k}_2}\hat{a}^\dagger_{\mathbf{k}_1}|0\rangle = |\mathbf{k}_2, \mathbf{k}_1\rangle \ . \tag{2.94}$$

The normalization of $|\mathbf{k}_1, \mathbf{k}_2\rangle$ is

$$\begin{aligned}\langle\mathbf{k}_1, \mathbf{k}_2|\mathbf{k}'_1, \mathbf{k}'_2\rangle &= \langle 0|\hat{a}_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}'_1}\hat{a}^\dagger_{\mathbf{k}'_2}|0\rangle = \langle 0|\hat{a}_{\mathbf{k}_2}[\hat{a}_{\mathbf{k}_1}, \hat{a}^\dagger_{\mathbf{k}'_1}\hat{a}^\dagger_{\mathbf{k}'_2}]|0\rangle \\ &= \langle 0|\hat{a}_{\mathbf{k}_2}[\hat{a}_{\mathbf{k}_1}, \hat{a}^\dagger_{\mathbf{k}'_1}]\hat{a}^\dagger_{\mathbf{k}'_2} + \hat{a}_{\mathbf{k}_2}\hat{a}^\dagger_{\mathbf{k}'_1}[\hat{a}_{\mathbf{k}_1}, \hat{a}^\dagger_{\mathbf{k}'_2}]|0\rangle \\ &= \delta^3(\mathbf{k}_1 - \mathbf{k}'_1)\delta^3(\mathbf{k}_2 - \mathbf{k}'_2) + \delta^3(\mathbf{k}_1 - \mathbf{k}'_2)\delta^3(\mathbf{k}_2 - \mathbf{k}'_1) \ .\end{aligned} \tag{2.95}$$

It is straightforward to extend these results to the general N particle state ;

$$|\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle = \hat{a}^\dagger_{\mathbf{k}_1}\cdots\hat{a}^\dagger_{\mathbf{k}_N}|0\rangle = |\mathbf{k}_1, \cdots, \underset{i}{\mathbf{k}_j}, \cdots, \underset{j}{\mathbf{k}_i}, \cdots, \mathbf{k}_N\rangle \ , \tag{2.96}$$

$$\hat{H}|\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle = \sum_{i=1}^{N}\omega_{k_i}|\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle \ , \ \ \hat{\mathbf{P}}|\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle = \sum_{i=1}^{N}\mathbf{k}_i|\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle \ , \tag{2.97}$$

$$\langle\mathbf{k}_1, \cdots, \mathbf{k}_N|\mathbf{k}'_1, \cdots, \mathbf{k}'_N\rangle = \sum_{\sigma(\text{permutation})}\delta^3(\mathbf{k}_1 - \mathbf{k}'_{\sigma_1})\cdots\delta^3(\mathbf{k}_N - \mathbf{k}'_{\sigma_N}) \ . \tag{2.98}$$

Thus, we find the remarkable property of the particle described by the scalar field $\hat{\phi}$;

State vector is completely symmetric under interchange of any set of two particles.

This property of the particle is called "Bose-Einstein statistics". The particle subject to this statistics is called "boson".

Problem (2.3-1) Prove the equal time commutabilities

$$[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{y}, t)] = 0 \ , \qquad [\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)] = 0 \ , \tag{2.99}$$

based on the commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}') \ , \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^{\dagger}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = 0 \ . \tag{2.100}$$

Also derive the following expression of $\hat{\mathbf{P}}$ by carrying out the space integration.

$$\hat{\mathbf{P}} = \int d^3\mathbf{k} \frac{\mathbf{k}}{2} (\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k}}) \ . \tag{2.101}$$

Problem (2.3-2) Prove the N particle state vector (2.96) satisfies the ortho-normality condition

$$\langle \mathbf{k}'_1, \mathbf{k}'_2, \cdots, \mathbf{k}'_N | \mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_N \rangle = \sum_{\sigma(\text{permutation})} \delta^3(\mathbf{k}'_1 - \mathbf{k}_{\sigma_1}) \cdots \delta^3(\mathbf{k}'_N - \mathbf{k}_{\sigma_N}) \ . \tag{2.102}$$

The general N-particle state $|\psi_{(N)}\rangle$ is expressed in terms of this vector as

$$|\psi_{(N)}\rangle = \frac{1}{\sqrt{N!}} \int d^3\mathbf{k}_1 \cdots d^3\mathbf{k}_N |\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle \ c(\mathbf{k}_1, \cdots, \mathbf{k}_N) \ , \tag{2.103}$$

where $c(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ is a completely symmetric function of $\mathbf{k}_i$ $(i = 1, \cdots, N)$. Show

$$\langle \psi_{(N)} | \psi_{(N)} \rangle = \int d^3\mathbf{k}_1 \cdots d^3\mathbf{k}_N \ c^*(\mathbf{k}_1, \cdots, \mathbf{k}_N) \ c(\mathbf{k}_1, \cdots, \mathbf{k}_N) \ . \tag{2.104}$$

Problem (2.3-3) Prove that the identity operator of the quantum states is expressed as

$$\hat{1} = |0\rangle\langle 0| + \sum_{N=1}^{\infty} \frac{1}{N!} \int d^3\mathbf{k}_1 \cdots d^3\mathbf{k}_N \ |\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_N\rangle\langle \mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_N| \ , \tag{2.105}$$

by showing, for any state $|\psi\rangle$, the identity equation

$$\hat{1}|\psi\rangle = |\psi\rangle \ . \tag{2.106}$$

Problem (2.3-4) Suppose the real scalar field is interacting with the classical matter distribution density $\rho(\mathbf{x})$. Hamiltonian is given by

$$\hat{H} = \int d^3\mathbf{x} \left(\frac{1}{2}\hat{\pi}(\mathbf{x})^2 + \frac{1}{2}(\nabla\hat{\phi}(\mathbf{x}))^2 + \frac{1}{2}m^2\hat{\phi}(\mathbf{x})^2 + g\rho(\mathbf{x})\hat{\phi}(\mathbf{x}) \right) \ . \tag{2.107}$$

Let us define the unitary operator $\hat{U}$ by

$$\hat{U} = e^{ig \int d^3\mathbf{x} d^3\mathbf{y} \hat{\pi}(\mathbf{x}) f(\mathbf{x}-\mathbf{y}) \rho(\mathbf{y})} \ , \tag{2.108}$$

$$f(\mathbf{x} - \mathbf{y}) \equiv \frac{1}{4\pi} \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \quad ; \ (-\nabla^2 + m^2) f(\mathbf{x} - \mathbf{y}) = \delta^3(\mathbf{x} - \mathbf{y}) \ . \tag{2.109}$$

Check the following unitary transformations of $\hat{\phi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$;

$$\hat{U}\hat{\phi}(\mathbf{x})\hat{U}^{-1} = \hat{\phi}(\mathbf{x}) + g \int d^3\mathbf{y} f(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) \ , \tag{2.110}$$

$$\hat{U}\hat{\pi}(\mathbf{x})\hat{U}^{-1} = \hat{\pi}(\mathbf{x}) \ . \tag{2.111}$$

According to this fact, show that the eigenstate $|n\rangle$ of $\hat{H}$ ($\hat{H}|n\rangle = E_n|n\rangle$) is represented in terms of the eigenstate $|n\rangle_0$ of the free scalar field ($\hat{H}_0|n\rangle_0 = E_n^0|n\rangle_0$) described by the Hamiltonian

$$\hat{H}_0 = \int d^3\mathbf{x} \left(\frac{1}{2}\hat{\pi}(\mathbf{x})^2 + \frac{1}{2}(\nabla\hat{\phi}(\mathbf{x}))^2 + \frac{1}{2}m^2\hat{\phi}(\mathbf{x})^2 \right) \tag{2.112}$$

as

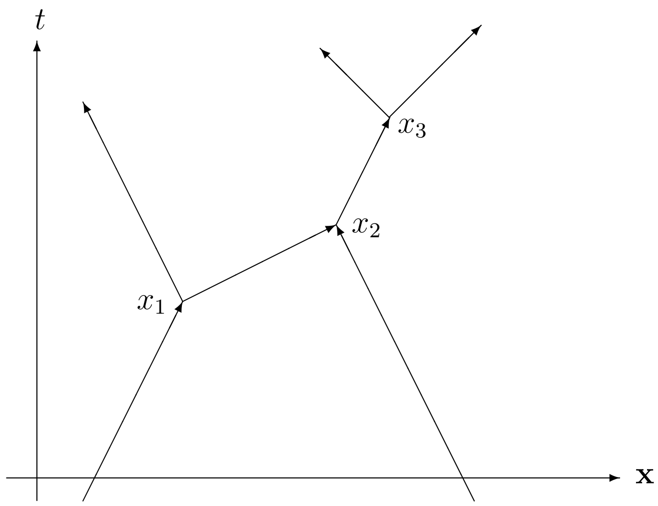
$$|n\rangle = \hat{U}|n\rangle_0 \tag{2.113}$$

$$E_n = E_n^0 - \frac{g^2}{2}\int d^3\mathbf{x}d^3\mathbf{y}\rho(\mathbf{x})f(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}) \ . \tag{2.114}$$

Derive the expectation value of $\hat{\phi}(\mathbf{x})$ with respect to the ground state $|0\rangle$ of $\hat{H}$.

2.4 T-Product and Propagator

T-Product : Imagine that there is an interaction between particles. Interactions of particles are described by the annihilations and creations of particles at various space-time points x_i. In quantum field theory, this is realized by the operation of the products of various operators $\hat{O}(x_i)$ on the state vector $|\psi\rangle$. In general, operators are non-commutative, $[\hat{O}(x_i), \hat{O}(x_j)] \neq 0$. Therefore, the ordering of the operators, which act on the state vector, is significant.



Intuitively, the natural ordering will be

Each operator $\hat{O}_i(x_i)$ stands from right to left along the ordering of each time x_i^0.

This is called T-product (time-ordered product) ;

$$T[\hat{O}_1(x_1)\cdots\hat{O}_n(x_n)] \ . \tag{2.115}$$

For example,

$$\begin{aligned} T[\hat{O}_1(x_1)\hat{O}_2(x_2)] &= \begin{cases} \hat{O}_1(x_1)\hat{O}_2(x_2) & x_1^0 > x_2^0 \\ \hat{O}_2(x_2)\hat{O}_1(x_1) & x_2^0 > x_1^0 \end{cases} \\ &= \theta(x_1^0 - x_2^0)\hat{O}_1(x_1)\hat{O}_2(x_2) + \theta(x_2^0 - x_1^0)\hat{O}_2(x_2)\hat{O}_1(x_1) \ , \end{aligned} \tag{2.116}$$

where $\theta(x^0)$ is the Heviside's θ function ;

$$\theta(x^0) \equiv \begin{cases} 1 & x^0 > 0 \\ 0 & x^0 < 0 \end{cases} . \tag{2.117}$$

Propagator : Let us express the field operator (2.66) in slightly compact form

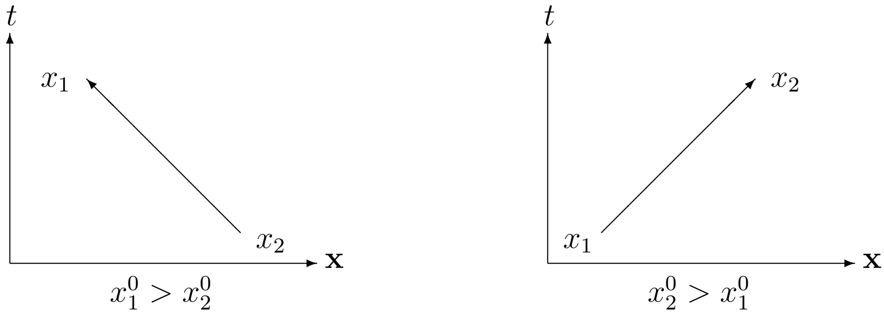
$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}^\dagger_{\mathbf{k}} e^{ik\cdot x} \right) , \tag{2.118}$$

$$k\cdot x \equiv k_\mu x^\mu = k^0 x^0 - \mathbf{k}\cdot\mathbf{x} , \quad k^\mu = (k^0 = \omega_k, \mathbf{k}) , \quad \omega_k = \sqrt{\mathbf{k}^2 + m^2} , \tag{2.119}$$

and take the matrix element of $T[\hat{\phi}(x_1)\hat{\phi}(x_2)]$ between vacuum $\langle 0|$ and $|0\rangle$;

$$\langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)]|0\rangle \equiv i\Delta_F(x_1 - x_2) . \tag{2.120}$$

The functon $\Delta_F(x_1 - x_2)$ is called Feynman propagator. Due to the property of the vacuum $\hat{a}|0\rangle = 0$ and $\langle 0|\hat{a}^\dagger = 0$, $\Delta_F(x_1 - x_2)$ expresses the amplitude of the process where the particle is created at a space-time point x_2 and annihilated at x_1 when $x_1^0 > x_2^0$, and particle is created at x_1 and annihilated at x_2 when $x_2^0 > x_1^0$.



Let us derive the explicit form of $\Delta_F(x_1 - x_2)$:

$$\begin{aligned} i\Delta_F(x_1 - x_2) &= \theta(x_1^0 - x_2^0)\langle 0| \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \hat{a}_{\mathbf{k}} e^{-ik\cdot x_1} \int \frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{k'}}} \hat{a}^\dagger_{\mathbf{k}'} e^{ik'\cdot x_2} |0\rangle \\ &+ \theta(x_2^0 - x_1^0)\langle 0| \int \frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{k'}}} \hat{a}_{\mathbf{k}'} e^{-ik'\cdot x_2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \hat{a}^\dagger_{\mathbf{k}} e^{ik\cdot x_1} |0\rangle \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_k} \left(\theta(x_1^0 - x_2^0) e^{-ik\cdot(x_1 - x_2)} + \theta(x_2^0 - x_1^0) e^{ik\cdot(x_1 - x_2)} \right) . \end{aligned} \tag{2.121}$$

We show that this is expressed in a compact form as a 4-dimensional integration

$$i\Delta_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x_1 - x_2)} , \quad k^2 \equiv (k^0)^2 - \mathbf{k}^2 , \quad d^4k \equiv dk^0 d^3\mathbf{k} , \tag{2.122}$$

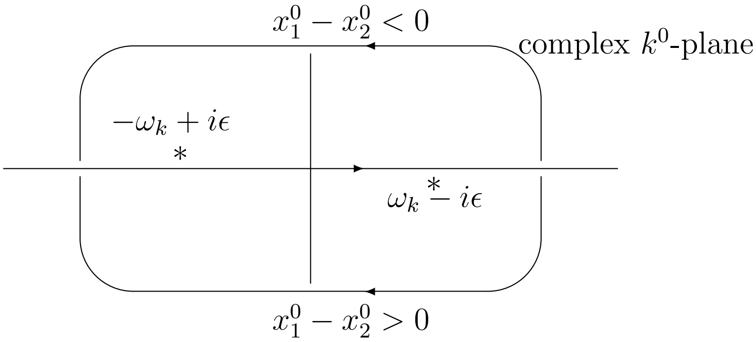
where ϵ is a positive infinitesimal quantity. Be careful that k^0 is an integration variable. Since

$$k^2 - m^2 = (k^0)^2 - \mathbf{k}^2 - m^2 = (k^0 - \omega_k)(k^0 + \omega_k) , \tag{2.123}$$

the right-hand-side is expressed as

$$\text{r.h.s.} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{i}{(k^0 - \omega_k)(k^0 + \omega_k) + i\epsilon} e^{-ik^0(x_1^0 - x_2^0)} \right] e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} . \tag{2.124}$$

Due to the presence of the factor $\exp(-ik^0(x_1^0 - x_2^0))$, when $x_1^0 - x_2^0 < 0$, the integration over k^0 can be extended to the closed contour circling from $+\infty$ to $-\infty$ on the upper complex k^0 plane, where $\exp(-ik^0(x_1^0 - x_2^0))$ vanishes.



The contour encloses the pole at $k^0 = -\omega_k + i\epsilon$. According to the residue theorem, the k^0 integral gives

$$[\cdots] = 2\pi i \frac{1}{2\pi} \frac{i}{-2\omega_k} e^{i\omega_k(x_1^0 - x_2^0)} \ . \tag{2.125}$$

Replacing in (2.124) the integration variable $\mathbf{k}$ by $-\mathbf{k}$, this gives the second term of $i\Delta_F(x_1 - x_2)$. When $x_1^0 - x_2^0 > 0$, the k^0 integration contour can be taken to circle the lower complex k^0 plane, which encloses the pole at $k^0 = \omega_k - i\epsilon$. Taking account of the minus sign due to the opposite integration direction, the k^0 integral gives this time

$$[\cdots] = -2\pi i \frac{1}{2\pi} \frac{i}{2\omega_k} e^{-i\omega_k(x_1^0 - x_2^0)} \ , \tag{2.126}$$

which reproduces the first term of $i\Delta_F(x_1 - x_2)$.

Thus, we have proved the formula

$$\Delta_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x_1 - x_2)} \ . \tag{2.127}$$

$\Delta_F(x)$ satisfies the following formulas ;

$$\Delta_F(x) = \Delta_F(-x) \ , \tag{2.128}$$

$$(\Box + m^2)\Delta_F(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-k\cdot x} = -\delta(x^0)\delta^3(\mathbf{x}) \equiv -\delta^4(x) \ . \tag{2.129}$$

Problem (2.4-1) Show that the 4-dimensional (non-equal-time) commutator of the free scalar field $\hat{\phi}(x)$ is represented in the form

$$[\hat{\phi}(x_1), \hat{\phi}(x_2)] \equiv i\Delta(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2)\epsilon(k^0) e^{-ik\cdot(x_1 - x_2)} \ , \tag{2.130}$$

$$\epsilon(k^0) = \begin{cases} +1 & k^0 > 0 \\ -1 & k^0 < 0 \end{cases} \ . \tag{2.131}$$

Next, replace all of the integration variable k^μ of this expression with k'^μ and define the new k^μ in terms of the Lorentz transformation matrix $\Lambda^\mu{}_\nu$ by $k^\mu = \Lambda^\mu{}_\nu k'^\nu$. Evidently, we have $k^2 = k'^2$ and $d^4k = d^4k'$. Derive the expression

$$i\Delta(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2)\epsilon(k'^0) e^{-ik\cdot(x_1' - x_2')} \ , \tag{2.132}$$

where $x'_i \equiv \Lambda^\mu{}_\nu x^\nu$. Confirm that the sign of k^0 is preserved under Lorentz transformation when 4-vector k^μ is timelike ($k^2 > 0$) which is ensured by $\delta(k^2 - m^2)$. This verifies the Lorentz invariance of the function $\Delta(x_1 - x_2) = \Delta(x'_1 - x'_2)$. When the interval of x_1 and x_2 is spacelike, that is, $(x_1 - x_2)^2 < 0$, it is possible to Lorentz transform two points x_1 and x_2 to the equal time points, $x_1^0 = x_2^0$. The equal-time commutator was $[\hat{\phi}(\mathbf{x}_1, t), \hat{\phi}(\mathbf{x}_2, t)] = 0$. Thus the Lorentz invariance of $\Delta(x_1 - x_2)$ implies that the spacelikely separated two fields $\hat{\phi}(x_1), \hat{\phi}(x_2)$ are commutative, $[\hat{\phi}(x_1), \hat{\phi}(x_2)] = 0$. The consequence of this fact is that, any two operations separated by the spacelike interval give no influence on each other. This is a reflection of the fact that the causality is accurately maintained in the quantum field theory.

Problem (2.4-2) Show that the vacuum expectation value of the T-product of four free scalar fields is expressed as

$$\begin{aligned} &\langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)]|0\rangle \\ &= i\Delta_F(x_1 - x_2)i\Delta_F(x_3 - x_4) + i\Delta_F(x_1 - x_3)i\Delta_F(x_2 - x_4) + i\Delta_F(x_1 - x_4)i\Delta_F(x_2 - x_3) \; . \end{aligned} \tag{2.133}$$

2.5 Complex Scalar Field

Suppose there are two free real scalar fields $\phi_1(x)$ and $\phi_2(x)$ with the equal mass m. The Lagrangian is a sum of each Lagrangian ;

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) \; . \tag{2.134}$$

Each quantum field $\hat{\phi}_i$ $(i = 1, 2)$ satisfies the Klein-Gordon equation

$$(\Box + m^2)\hat{\phi}_i(x) = 0 \qquad i = 1, 2 \; . \tag{2.135}$$

Thus, they are expanded in terms of each creation and annihilation operators $\hat{a}_{i\mathbf{k}}$ and $\hat{a}^\dagger_{i\mathbf{k}}$;

$$\hat{\phi}_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(\hat{a}_{i\mathbf{k}}e^{-ik\cdot x} + \hat{a}^\dagger_{i\mathbf{k}}e^{ik\cdot x}\right) \; , \tag{2.136}$$

$$[\hat{a}_{i\mathbf{k}}, \hat{a}^\dagger_{j\mathbf{k}'}] = \delta_{ij}\delta^3(\mathbf{k} - \mathbf{k}') \; , \qquad \text{other } [*, *] = 0 \; . \tag{2.137}$$

Conserved Current and Charge : This system has a following remarkable property. Let us define the Lorentz vector operator $\hat{j}_\mu(x)$, called "current", by

$$\hat{j}_\mu(x) \equiv \hat{\phi}_2(x)\partial_\mu\hat{\phi}_1(x) - \hat{\phi}_1(x)\partial_\mu\hat{\phi}_2(x) \; . \tag{2.138}$$

Due to the Klein-Gordon equation, $\hat{j}_\mu(x)$ satisfies the continuity equation

$$\partial_\mu\hat{j}^\mu = \hat{\phi}_2\Box\hat{\phi}_1 - \hat{\phi}_1\Box\hat{\phi}_2 = -\hat{\phi}_2 m^2\hat{\phi}_1 + \hat{\phi}_1 m^2\hat{\phi}_2 = 0 \; . \tag{2.139}$$

The current $\hat{j}^\mu$ which satisfies $\partial_\mu\hat{j}^\mu = 0$ is in general called conserved current. The space integral of its time component $\hat{j}^0$ is called charge $\hat{Q}$;

$$\hat{Q} = \int d^3\mathbf{x}\; \hat{j}^0(x) = \int d^3\mathbf{x}\left(\hat{\phi}_2\dot{\hat{\phi}}_1 - \hat{\phi}_1\dot{\hat{\phi}}_2\right) = i\int d^3\mathbf{k}\left(\hat{a}^\dagger_{1\mathbf{k}}\hat{a}_{2\mathbf{k}} - \hat{a}^\dagger_{2\mathbf{k}}\hat{a}_{1\mathbf{k}}\right) \; . \tag{2.140}$$

Due to the current conservation, $\hat{Q}$ is a conserved quantity :

$$\dot{\hat{Q}} = \int d^3\mathbf{x} \, \frac{\partial}{\partial t} \hat{j}^0(x) = -\int d^3\mathbf{x} \, \nabla \cdot \hat{\mathbf{j}} = -\int_{|\mathbf{x}| \to \infty} d^2\mathbf{S} \cdot \hat{\mathbf{j}} = 0 \tag{2.141}$$

Another Viewpoint : Let us combine ϕ_1 and ϕ_2 to form a complex field ϕ ;

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \ , \qquad \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \ . \tag{2.142}$$

Then, the quantum field $\hat{\phi}$ is expressed as

$$\hat{\phi}(x) = \frac{1}{\sqrt{2}}(\hat{\phi}_1 + i\hat{\phi}_2(x)) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_{(+)\mathbf{k}} e^{-ik\cdot x} + \hat{a}^\dagger_{(-)\mathbf{k}} e^{ik\cdot x} \right) \ , \tag{2.143}$$

$$\hat{\phi}^\dagger(x) = \frac{1}{\sqrt{2}}(\hat{\phi}_1 - i\hat{\phi}_2(x)) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_{(-)\mathbf{k}} e^{-ik\cdot x} + \hat{a}^\dagger_{(+)\mathbf{k}} e^{ik\cdot x} \right) \ , \tag{2.144}$$

where

$$\hat{a}_{(\pm)\mathbf{k}} \equiv \frac{1}{\sqrt{2}}(\hat{a}_{1\mathbf{k}} \pm i\hat{a}_{2\mathbf{k}}) \ , \qquad \hat{a}^\dagger_{(\pm)\mathbf{k}} \equiv \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{1\mathbf{k}} \mp i\hat{a}^\dagger_{2\mathbf{k}}) \ . \tag{2.145}$$

They satisfy the commutation relations

$$[\hat{a}_{(\pm)\mathbf{k}}, \hat{a}^\dagger_{(\pm)\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}') \ , \tag{2.146}$$

$$[\hat{a}_{(\pm)\mathbf{k}}, \hat{a}^\dagger_{(\mp)\mathbf{k}'}] = 0 \ , \quad [\hat{a}, \hat{a}] = 0 \ , \quad [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \ . \tag{2.147}$$

The Hamiltonian $\hat{H}$, momentum $\hat{\mathbf{P}}$ and charge $\hat{Q}$ turn out to be simply a sum of two sectors $(+)$ and $(-)$;

$$\hat{H} = \int d^3\mathbf{k} \, \omega_k \left(\hat{a}^\dagger_{(+)\mathbf{k}} \hat{a}_{(+)\mathbf{k}} + \hat{a}^\dagger_{(-)\mathbf{k}} \hat{a}_{(-)\mathbf{k}} \right) \tag{2.148}$$

$$\hat{\mathbf{P}} = \int d^3\mathbf{k} \, \mathbf{k} \left(\hat{a}^\dagger_{(+)\mathbf{k}} \hat{a}_{(+)\mathbf{k}} + \hat{a}^\dagger_{(-)\mathbf{k}} \hat{a}_{(-)\mathbf{k}} \right) \tag{2.149}$$

$$\hat{Q} = \int d^3\mathbf{k} \, \left(\hat{a}^\dagger_{(+)\mathbf{k}} \hat{a}_{(+)\mathbf{k}} - \hat{a}^\dagger_{(-)\mathbf{k}} \hat{a}_{(-)\mathbf{k}} \right) \tag{2.150}$$

$$\tag{2.151}$$

The commutators of $\hat{Q}$ with $\hat{a}_{(\pm)\mathbf{k}}$ and $\hat{a}^\dagger_{(\pm)\mathbf{k}}$ are

$$[\hat{Q}, \hat{a}_{(\pm)\mathbf{k}}] = \mp \hat{a}_{(\pm)\mathbf{k}} \ , \quad [\hat{Q}, \hat{a}^\dagger_{(\pm)\mathbf{k}}] = \pm \hat{a}^\dagger_{(\pm)\mathbf{k}} \ . \tag{2.152}$$

Thus, the operators $\hat{a}_{(\pm)}$ annihilate and $\hat{a}^\dagger_{(\pm)}$ create a particle with charge ± 1. If we call the particle created by $\hat{a}^\dagger_{(+)}$ "particle", the particle created by $\hat{a}^\dagger_{(-)}$ is called "antiparticle" :

From (9.49), (2.144) and (2.152), we find

$$[\hat{Q}, \hat{\phi}] = -\hat{\phi} \ , \qquad [\hat{Q}, \hat{\phi}^\dagger] = \hat{\phi}^\dagger \ . \tag{2.153}$$

These commutators show that, $\hat{\phi}$ operated on a state $|\psi\rangle$ decreses the charge of the state by one unit by annihilating a particle or creating an antiparticle. $\hat{\phi}^\dagger$ works in a contrary way.

Fock Space : The vacuum $|0\rangle$ is originally defined by $\hat{a}_{1\mathbf{k}}|0\rangle = \hat{a}_{2\mathbf{k}}|0\rangle = 0$. Thus,

$$\hat{a}_{(+)\mathbf{k}}|0\rangle = \hat{a}_{(-)\mathbf{k}}|0\rangle = 0 \ . \tag{2.154}$$

The one-particle state is

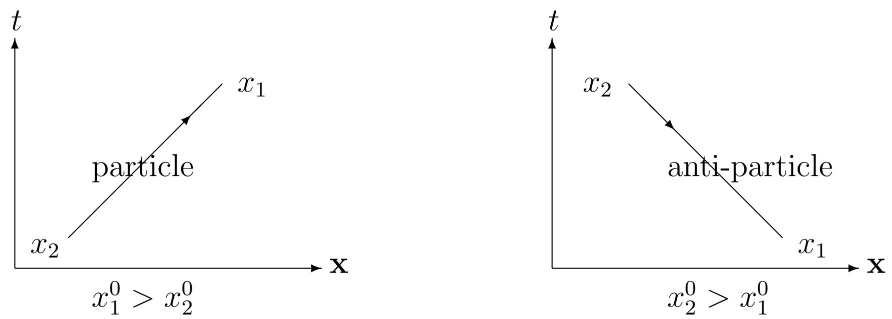
$$\hat{a}^\dagger_{(+)\mathbf{k}}|0\rangle \qquad \text{particle with momentum } \mathbf{k} \ , \tag{2.155}$$

$$\hat{a}^\dagger_{(-)\mathbf{k}}|0\rangle \qquad \text{anti-particle with momentum } \mathbf{k} \ . \tag{2.156}$$

Propagator : Let us first examine $\langle 0|T[\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2)]|0\rangle$;

$$\begin{aligned}
\langle 0|T[\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2)]|0\rangle &= \frac{1}{2}\langle 0|T[\Big(\hat{\phi}_1(x_1)+i\hat{\phi}_2(x_1)\Big)\Big(\hat{\phi}_1^\dagger(x_2)-i\hat{\phi}_2^\dagger(x_2)\Big)]|0\rangle \\
&= \frac{1}{2}\langle 0|T[\Big(\hat{\phi}_1(x_1)\hat{\phi}_1^\dagger(x_2)+\hat{\phi}_2(x_1)\hat{\phi}_2^\dagger(x_2)\Big)]|0\rangle \\
&= i\Delta_F(x_1-x_2)
\end{aligned} \tag{2.157}$$

This propagator represents the processes where, a particle is created at x_2 and annihilated at x_1 when $x_1^0 > x_2^0$, and an anti-particle is created at x_1 and annihilated at x_2 when $x_2^0 > x_1^0$. When figuring these processes, we add an arrow pointing from $\hat{\phi}^\dagger(x_2)$ to $\hat{\phi}(x_1)$. Then the particle moves along the arrow but the antiparticle moves opposite to the arrow.



Owing to the charge conservation, other propagators vanish ;

$$\langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)]|0\rangle = \langle 0|T[\hat{\phi}^\dagger(x_1)\hat{\phi}^\dagger(x_2)]|0\rangle = 0 \ . \tag{2.158}$$

Problem (2.5-1) Derive the final expression of $\hat{Q}$ presented in (2.140)

$$\hat{Q} = i\int d^3\mathbf{k}\left(\hat{a}^\dagger_{1\mathbf{k}}\hat{a}_{2\mathbf{k}} - \hat{a}^\dagger_{2\mathbf{k}}\hat{a}_{1\mathbf{k}}\right) \ . \tag{2.159}$$

by carrying out the space integration.

Problem (2.5-2) Check the results (2.158) based on the real fields $\hat{\phi}_1$ and $\hat{\phi}_2$.

2.6 Chapter-Project/Q

We have completed the framework for the description of the motion of the free quantum scalar fields. Though it treats only free scalar fields, the framework is applicable to the quantum fields interacting with the classical external source as we found in the problem **(2.3-4)**. Here, we slightly modify the model and extract physically significant information on the particle creation through the interaction with the classical source.

Suppose the real scalar field $\hat{\phi}(\mathbf{x})$ interacts, for the time interval $0 < t < T$, with the classical matter density $\rho(\mathbf{x})$ through the interaction Hamiltonian

$$\hat{H}_{\rm int} = \int d^3\mathbf{x}\ g\rho(\mathbf{x})\hat{\phi}(\mathbf{x}) = \int d^3\mathbf{k}\ \omega_k(f^*_\mathbf{k}\hat{a}_\mathbf{k} + f_\mathbf{k}\hat{a}^\dagger_\mathbf{k}) \ : \qquad f_\mathbf{k} = \frac{g}{\omega_k\sqrt{2\omega_k}}\int d^3\mathbf{x}\ \rho(\mathbf{x})\frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \ , \tag{2.160}$$

where g represents the strength of the interaction. We treat this system in the Schrödinger picture. The total Hamiltonian $\hat{H}$ for $0 < t < T$ is a sum of the free Hamiltonian

$$\hat{H}_0 = \int d^3\mathbf{k}\ \omega_k \hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} \tag{2.161}$$

and the interaction Hamiltonian $\hat{H}_{\rm int}$:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int} = \int d^3\mathbf{k}\ \omega_k \left[(\hat{a}_\mathbf{k} + f_\mathbf{k})^\dagger (\hat{a}_\mathbf{k} + f_\mathbf{k}) - f^*_\mathbf{k} f_\mathbf{k} \right] \ , \qquad 0 < t < T \ . \tag{2.162}$$

Let us assume the state $|\psi(t)\rangle$ for $t < 0$ is the vacuum $|0\rangle$ of $\hat{H}_0$, and calculate the number of particles N_T created through the interaction. Since the system is free at $t > T$, the particle number operator $\hat{N}$ commutes with $\hat{H}_0$ and it is given by

$$\hat{N} = \int d^3\mathbf{k}\ \hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} \ : \qquad \hat{N} |\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle = N |\mathbf{k}_1, \cdots, \mathbf{k}_N\rangle \ . \tag{2.163}$$

Therefore, N_T evaluated by the state $|\psi(t > T)\rangle = e^{-\hat{H}_0(t-T)} e^{-i\hat{H}T} |0\rangle$ is expressed as

$$N_T = \langle \psi(t) | \hat{N} | \psi(t) \rangle = \langle 0 | e^{i\hat{H}T} \hat{N} e^{-i\hat{H}T} | 0 \rangle \ , \qquad t > T \ . \tag{2.164}$$

For the calculation of the matrix element (2.164), it is useful to introduce the unitary operator

$$\hat{U} = \exp - \int d^3\mathbf{k}\ (f^*_\mathbf{k} \hat{a}_\mathbf{k} - f_\mathbf{k} \hat{a}^\dagger_\mathbf{k}) \ : \qquad \hat{U}^\dagger = \hat{U}^{-1} \tag{2.165}$$

which realizes, through the similarity transformation formula (2.36),

$$\hat{U}^{-1} \hat{a}_\mathbf{k} \hat{U} = \hat{a}_\mathbf{k} + f_\mathbf{k} \qquad \text{and then} \qquad (\hat{a}_\mathbf{k} + f_\mathbf{k})^\dagger (\hat{a}_\mathbf{k} + f_\mathbf{k}) = \hat{U}^{-1} \hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} \hat{U} \ . \tag{2.166}$$

Thus, we have

$$\hat{H} = \hat{U}^{-1} \hat{H}_0 \hat{U} - \int d^3\mathbf{k}\ \omega_k f^*_\mathbf{k} f_\mathbf{k} \qquad \text{and} \qquad \hat{U} \hat{N} \hat{U}^{-1} = \int d^3\mathbf{k}\ (\hat{a}_\mathbf{k} - f_\mathbf{k})^\dagger (\hat{a}_\mathbf{k} - f_\mathbf{k}) \ . \tag{2.167}$$

The second term in $\hat{H}$ cancels in (2.164). Consequently, N_T is expressed as

$$\begin{aligned} N_T &= \int d^3\mathbf{k}\ \langle 0 | \hat{U}^{-1} e^{i\hat{H}_0 T} \hat{U} \hat{N} \hat{U}^{-1} e^{-i\hat{H}_0 T} \hat{U} | 0 \rangle \\ &= \int d^3\mathbf{k}\ \langle 0 | \hat{U}^{-1} e^{i\hat{H}_0 T} (\hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} - f^*_\mathbf{k} \hat{a}_\mathbf{k} - f_\mathbf{k} \hat{a}^\dagger_\mathbf{k} + f^*_\mathbf{k} f_\mathbf{k}) e^{-i\hat{H}_0 T} \hat{U} | 0 \rangle \\ &= \int d^3\mathbf{k}\ \langle 0 | \hat{U}^{-1} (\hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} - f^*_\mathbf{k} \hat{a}_\mathbf{k} e^{-i\omega_k T} - f_\mathbf{k} \hat{a}^\dagger_\mathbf{k} e^{i\omega_k T} + f^*_\mathbf{k} f_\mathbf{k}) \hat{U} | 0 \rangle \\ &= \int d^3\mathbf{k}\ \langle 0 | \left((\hat{a}_\mathbf{k} + f_\mathbf{k})^\dagger (\hat{a}_\mathbf{k} + f_\mathbf{k}) - f^*_\mathbf{k} (\hat{a}_\mathbf{k} + f_\mathbf{k}) e^{-i\omega_k T} - f_\mathbf{k} (\hat{a}_\mathbf{k} + f_\mathbf{k})^\dagger e^{i\omega_k T} + f^*_\mathbf{k} f_\mathbf{k} \right) | 0 \rangle \ . \end{aligned} \tag{2.168}$$

Taking the matrix elements, we obtain the formula

$$N_T = \int d^3\mathbf{k}\ 2 f^*_\mathbf{k} f_\mathbf{k} \left(1 - \cos \omega_k T\right) \ . \tag{2.169}$$

From this expression, we understand that in the limit $T \to \infty$, N_T approaces to the value

$$N_{T\to\infty} = \int d^3\mathbf{k}\ 2f^*_{\mathbf{k}}f_{\mathbf{k}} \tag{2.170}$$

because the second term vanishes in this limit due to the rapid sign change of the integrand. This situation represents the limit where the number of particles created from the source is balanced with the number of particles absorbed by the source.

Let us examine the more detailed analysis. Suppose the matter distribution in (2.160) is the normalized Gaussian function

$$\rho(\mathbf{x}) = \frac{1}{(\sqrt{\pi}D)^3}e^{-\mathbf{x}^2/D^2} \quad : \qquad \int d^3\mathbf{x}\ \rho(\mathbf{x}) = 1\ . \tag{2.171}$$

Then, the Fourier component $f_{\mathbf{k}}$ is given by

$$f_{\mathbf{k}} = \frac{g}{\omega_k\sqrt{2\omega_k}(2\pi)^{3/2}}\ e^{-D^2\mathbf{k}^2/4} \tag{2.172}$$

and N_T is expressed as

$$N_T = g^2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1-\cos\omega_k T}{\omega_k^3}\ e^{-D^2\mathbf{k}^2/2}\ . \tag{2.173}$$

In the case of massless scalar field ($m = 0$), it takes a compact expression

$$N_T(m=0) = \frac{g^2}{2\pi^2}\int_0^\infty dx\ \frac{1-\cos rx}{x}e^{-x^2} \quad : \qquad r \equiv \frac{\sqrt{2}T}{D}\ . \tag{2.174}$$

This is a monotonically increasing function of T, and it diverges in the limit $T \to \infty$.

If we limit the interaction time T to an infinitesimal time interval ΔT, $N_{\Delta T}$ is expressed as

$$N_{\Delta T} = (\Delta T)^2\frac{g^2}{2\pi^2D^2}\int_0^\infty dx\ \frac{x^2}{\sqrt{x^2+m^2D^2/2}}\ e^{-x^2} + O\left((\Delta T)^4\right)\ . \tag{2.175}$$

This diverges in the localized limit $D \to 0$ of $\rho(\mathbf{x})$. In the case of massless scalar field, we obtain

$$N_{\Delta T} = (\Delta T)^2\frac{g^2}{4\pi^2D^2} + O\left((\Delta T)^4\right)\ . \tag{2.176}$$

Chapter 3

Free Dirac Quantum Field

3.1 Statistics of Particles and Commutators of Fields

Quantum Statistics : Quantum mechanics gives a stringent constraint on the quantum state containing the same kind of particles, which is called quantum statistics. The allowed statistics is only two kind :

Bose-Einstein statistics :	State vector is completely symmetric under interchange of particles.	: Boson
Fermi-Dirac statistics :	State vector is completely anti-symmetric under interchange of particles.	: Fermion

Description of Particles : The description of particles in quantum field theory is based on the vacuum $|0\rangle$ which serves, by a requirement $\hat{a}_{\mathbf{k}}|0\rangle = 0$, a ground state of the Hamiltonian

$$\hat{H} = \int d^3\mathbf{k}\ \omega_k \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \ , \qquad \hat{H}|0\rangle = 0 \ , \tag{3.1}$$

and the commutation relations of $\hat{H}$ with the annihilation and creation operators

$$[\hat{H}, \hat{a}_{\mathbf{k}}] = -\omega_k \hat{a}_{\mathbf{k}} \ , \quad [\hat{H}, \hat{a}^\dagger_{\mathbf{k}}] = \omega_k \hat{a}^\dagger_{\mathbf{k}} \ . \tag{3.2}$$

The boson system realizes these commutation relations by

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}') \ , \ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0 \ , \ [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0 \ , \tag{3.3}$$

based on the identity relation

$$[\hat{a}^\dagger_{\mathbf{k}'} \hat{a}_{\mathbf{k}'}, \hat{a}_{\mathbf{k}}] = \hat{a}^\dagger_{\mathbf{k}'}[\hat{a}_{\mathbf{k}'}, \hat{a}_{\mathbf{k}}] + [\hat{a}^\dagger_{\mathbf{k}'}, \hat{a}_{\mathbf{k}}]\hat{a}_{\mathbf{k}'} \ . \tag{3.4}$$

Owing to the commutativity $\hat{a}^\dagger_{\mathbf{k}_i} \hat{a}^\dagger_{\mathbf{k}_j} = \hat{a}^\dagger_{\mathbf{k}_j} \hat{a}^\dagger_{\mathbf{k}_i}$, the multi-particle state $\hat{a}^\dagger_{\mathbf{k}_1} \cdots \hat{a}^\dagger_{\mathbf{k}_N}|0\rangle$ is completely symmetric under the interchange of any set of $\hat{a}^\dagger_{\mathbf{k}_i}$ and $\hat{a}^\dagger_{\mathbf{k}_j}$. The commutators (3.3) came from the commutators of the fields

$$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}) \ , \ [\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = 0 \ , \ [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0 \ . \tag{3.5}$$

The key to the description of fermion system is the identity relation alternative to (3.4), which is a “trivial miracle” of mathematics,

$$[\hat{a}^\dagger_{\mathbf{k}'} \hat{a}_{\mathbf{k}'}, \hat{a}_{\mathbf{k}}] = \hat{a}^\dagger_{\mathbf{k}'}\{\hat{a}_{\mathbf{k}'}, \hat{a}_{\mathbf{k}}\} - \{\hat{a}^\dagger_{\mathbf{k}'}, \hat{a}_{\mathbf{k}}\}\hat{a}_{\mathbf{k}'} \ , \tag{3.6}$$

where, $\{A, B\} \equiv AB + BA$ is called anti-commutator. We find the anticommutators

$$\{\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\} = \delta^3(\mathbf{k} - \mathbf{k}') \ , \quad \{\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}\} = 0 \ , \ \{\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\} = 0 \tag{3.7}$$

also realize (3.2). The anti-commutativity $\hat{a}^\dagger_{\mathbf{k}_i}\hat{a}^\dagger_{\mathbf{k}_j} = -\hat{a}^\dagger_{\mathbf{k}_j}\hat{a}^\dagger_{\mathbf{k}_i}$ ensures the complete anti-symmetry of the multi-particle state $\hat{a}^\dagger_{\mathbf{k}_1} \cdots \hat{a}^\dagger_{\mathbf{k}_N}|0\rangle$. Thus, the quantization prescription for fermionic field is

$$\{\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\} = i\delta^3(\mathbf{x} - \mathbf{y}) \ , \ \{\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})\} = 0 \ , \ \{\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\} = 0 \ . \tag{3.8}$$

Question : Can we quantize scalar field by fermi statistics ? The answer is No!! The Hamiltonian was

$$\hat{H} = \int d^3\mathbf{x} \left(\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 \right) = \int d^3\mathbf{k} \ \frac{\omega_k}{2} \left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} \right) \ . \tag{3.9}$$

The anti-commutator leads to $\hat{H} = \int d^3\mathbf{k} \ \frac{1}{2}\omega_k \delta^3(\mathbf{k} - \mathbf{k}) = [\text{constant}]$. Thus, $[\hat{H}, \hat{\phi}] = 0$, $[\hat{H}, \hat{\pi}] = 0$, and $\hat{H}$ cannot work as a quantum mechanical operator. Therefore, the scalar particle, which has no spin degree of freedom, must be subject to the Bose-Einstein statics.

3.2 Dirac Equation

The Dirac spinor field $\psi(x)$ is a 4-component complex field ;

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \ . \tag{3.10}$$

Whenever the field $\psi(x)$ represents a particle with mass m, its energy ω_k and momentum $\mathbf{k}$ should satisfy the Einstein's relation $\omega_k^2 - \mathbf{k}^2 = m^2$. Therefore, $\psi(x)$ must be subject to the Klein-Gordon equation $(\Box + m^2)\psi(x) = 0$. If we assume that the Klein-Gordon equation is realized as the Euler-Lagrange equation of $\psi(x)$, its Lagrangian takes a form $\mathcal{L} = \partial_\mu\psi^\dagger\partial^\mu\psi - m^2\psi^\dagger\psi$. Since this Lagrangian merely represents four species of complex scalar fields, the resulting particles are bosons.

Dirac Equation : If $\psi(x)$ represents the particle subject to the Fermi-Dirac statistics, its equation of motion must be "stronger" than the Klein-Gordon Equation. It should be the first-order equation with respect to the derivative. The equation is called Dirac equation ;

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad ; \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \ , \tag{3.11}$$

where γ^μ ($\mu = 0, 1, 2, 3$) are the 4×4 constant matrices which act on the four components of $\psi(x)$. The term m also tacitly contains the 4×4 unit matrix 1. The condition for the solution of this equation to satisfy the Klein-Gordon equation is obtained by the requirement

$$\begin{aligned} 0 &= (-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\psi(x) \\ &= (\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2)\psi(x) \\ &= \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2 \right)\psi(x) \qquad \text{because} \ \ \partial_\mu\partial_\nu = \partial_\nu\partial_\mu \\ &\Rightarrow (\Box + m^2)\psi(x) \ . \end{aligned} \tag{3.12}$$

That is, the matrices γ^μ must satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot 1 \qquad : \ (\gamma^0)^2 = 1 \ , \ \ (\gamma^i)^2 = -1 \ \ (i = 1, 2, 3) \ . \tag{3.13}$$

For the later convenience, we require the (anti-)hermiticity

$$(\gamma^0)^\dagger = \gamma^0 \ , \ \ (\gamma^i)^\dagger = -\gamma^i \qquad : \ \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \ . \tag{3.14}$$

Standard Representation : Dirac composed γ^μ satisfying (3.13) and (3.14) in the form

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} \ , \ \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad i = 1, 2, 3 \ , \tag{3.15}$$

where σ^0 is a 2×2 unit matrix, and σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{3.16}$$

It will be an instructive exercise to confirm that there is no possibility for the smaller matrices than 4×4 ones to satisfy (3.13) and (3.14).

Unitary Equivalence : Suppose the two sets of gamma matrices γ^μ and $\tilde{\gamma}^\mu$ both satisfy the conditions (3.13) and (3.14). Then, they are always connected by the unitary transformation

$$\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger \ , \qquad UU^\dagger = 1 \ . \tag{3.17}$$

We skip here the proof of this Pauli's "fundamental theorem" which is tedious though straightforward. The theorem (3.17) states that , when $\psi(x)$ is the solution of Dirac equation with γ^μ, the solution with $\tilde{\gamma}^\mu$ is given by $\tilde{\psi}(x) \equiv U\psi(x)$. That is, $\psi(x)$ and $\tilde{\psi}(x)$ represent the "same" physical situation using "different" convention for the labeling of the four components of $\psi(x)$.

Lorentz Transformation of $\psi(x)$: Under the Lorentz transformation $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $\psi(x)$ should also recieve a transformation

$$\psi(x) \underset{\text{L.T.}}{\longrightarrow} \psi'(x') = S\psi(x) \ . \tag{3.18}$$

S is the 4×4 transformation matrix, which we are going to investigate. The "principle of relativity" states that, the equation of motion takes the same form in any inertial frame. Therefore, when $\psi(x)$ satisfies the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$, $\psi'(x')$ should also satisfy the Dirac equation

$$(i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0 \ , \qquad \partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} \ . \tag{3.19}$$

Using the relation (1.27), $\partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu$, we obtain

$$(i\gamma^\mu\partial'_\mu - m)\psi'(x') = S(iS^{-1}\gamma^\mu S\Lambda_\mu{}^\nu\partial_\nu - m)\psi(x) \ . \tag{3.20}$$

Thus, only when S satisfies the requirement

$$S^{-1}\gamma^\mu S\Lambda_\mu{}^\nu = \gamma^\nu \ , \tag{3.21}$$

(3.19) is realized ;

$$(i\gamma^\mu\partial'_\mu - m)\psi'(x') = S(i\gamma^\nu\partial_\nu - m)\psi(x) = 0 \ . \tag{3.22}$$

Owing to the condition of the Lorentz transformation $\Lambda^{\mu}{}_{\rho}\Lambda_{\mu}{}^{\nu} = \delta^{\nu}_{\rho}$, (3.21) is expressed in the form

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\rho}\gamma^{\rho} \ . \tag{3.23}$$

The solution to this equation $S(\Lambda^{\mu}{}_{\nu})$ is uniquely determined in terms of $\Lambda^{\mu}{}_{\nu}$ under the requirement of the "group property" of the Lorentz transformation ;

$$S(\delta^{\mu}{}_{\nu}) = 1 \ , \qquad S(\Lambda^{\mu}_{2\,\rho}\Lambda^{\rho}_{1\,\nu}) = S(\Lambda^{\mu}_{2\,\nu})S(\Lambda^{\mu}_{1\,\nu}) \ . \tag{3.24}$$

The latter represents the multiplicative nature of two successive Lorentz transformations by $\Lambda^{\mu}_{1\,\nu}$ and $\Lambda^{\mu}_{2\,\nu}$. The four component quantity $\psi(x)$ which transforms according to (3.18) is called Lorentz spinor.

Taking a hermitian conjugate of (3.23), we have

$$S^{\dagger}\gamma^{\mu\dagger}(S^{-1})^{\dagger} = \Lambda^{\mu}{}_{\rho}\gamma^{\rho\dagger} \tag{3.25}$$

By the relations $\gamma^{\mu\dagger} = \gamma^0\gamma^{\mu}\gamma^0$ and $(\gamma^0)^2 = 1$, this is expressed as

$$(\gamma^0 S^{\dagger}\gamma^0)\gamma^{\mu}(\gamma^0 S^{\dagger}\gamma^0)^{-1} = \Lambda^{\mu}{}_{\rho}\gamma^{\rho} \ . \tag{3.26}$$

Comparing this expression with (3.23), we obtain, from the uniqueness of S, the impotant nature of the transformation matrix S :

$$\gamma^0 S^{\dagger}\gamma^0 = S^{-1} \ . \tag{3.27}$$

Dirac Conjugate : Let us define the four component row spinor by

$$\bar{\psi}(x) \equiv \psi^{\dagger}(x)\gamma^0 \ . \tag{3.28}$$

Under the Lorentz transformation, $\bar{\psi}(x)$ transforms as

$$\bar{\psi}(x) \underset{\text{L.T.}}{\longrightarrow} \bar{\psi}'(x') \equiv \psi'^{\dagger}(x')\gamma^0 = \psi^{\dagger}(x)S^{\dagger}\gamma^0 = \psi^{\dagger}(x)\gamma^0\gamma^0 S^{\dagger}\gamma^0 = \bar{\psi}(x)S^{-1} \ . \tag{3.29}$$

This determines the Lorentz transformation property of the bi-linear quantities :

$$\bar{\psi}\psi \underset{\text{L.T.}}{\longrightarrow} \bar{\psi}'\psi' = \bar{\psi}S^{-1}S\psi = \bar{\psi}\psi \quad : \ \text{Lorentz scalar} \tag{3.30}$$

$$\bar{\psi}\gamma^{\mu}\psi \underset{\text{L.T.}}{\longrightarrow} \bar{\psi}'\gamma^{\mu}\psi' = \bar{\psi}S^{-1}\gamma^{\mu}S\psi = \Lambda^{\mu}{}_{\rho}\bar{\psi}\gamma^{\rho}\psi \quad : \ \text{Lorentz vector} \tag{3.31}$$

Taking the dagger of the Dirac equation $(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0$, we have $\psi^{\dagger}(x)(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_{\mu} - m) = 0$, where $\overleftarrow{\partial}_{\mu}$ means that the differential operates on $\psi^{\dagger}(x)$. Inserting $\gamma^0\gamma^0 = 1$ just after $\psi^{\dagger}(x)$ and multiplying γ^0 from the right, we obtain the Dirac equation for $\bar{\psi}(x)$:

$$\bar{\psi}(x)(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu} - m) = 0 \ . \tag{3.32}$$

Problem (3.2-1) Let us express the infinitesimal version of the Lorentz transformation matrices $\Lambda^{\mu}{}_{\nu}$ and $\Lambda_{\mu}{}^{\nu}$ which satisfy the condition (1.16) as

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu} \ , \qquad \Lambda_{\mu}{}^{\nu} = \delta^{\nu}_{\mu} - \epsilon^{\nu}{}_{\mu} \ : \qquad |\epsilon^{\mu}{}_{\nu}| \ll 1 \ . \tag{3.33}$$

Show from the definition (1.13) of $\Lambda_\mu{}^\nu$ that

$$\epsilon_{\mu\nu} = g_{\mu\rho}\epsilon^\rho{}_\nu = -\epsilon_{\nu\mu} \tag{3.34}$$

is an antisymmetric matrix. Therefore, the matrix $\epsilon_{\mu\nu}$ contains six independent parameters. The three parameters ϵ_{ij} $(i,j = 1,2,3)$ represent the three-dimensional space rotation and remaining three parameters ϵ_{0i} $(i = 1,2,3)$ represent the Lorentz boost along the x^i-axis.

Let us represent the corresponding transformation matrix S for the Lorentz spinor as

$$S = 1 - \frac{i}{4}\sigma^{\alpha\beta}\epsilon_{\alpha\beta}\ , \qquad S^{-1} = 1 + \frac{i}{4}\sigma^{\alpha\beta}\epsilon_{\alpha\beta}\ , \tag{3.35}$$

where $\sigma^{\alpha\beta} = -\sigma^{\beta\alpha}$ is the yet unknowen 4×4 matrix. From the infinitesimal version of the condition (3.23), show that the commutator of $\sigma^{\alpha\beta}$ must realize

$$[\sigma^{\alpha\beta}, \gamma^\mu] = 2i(\gamma^\alpha g^{\mu\beta} - \gamma^\beta g^{\mu\alpha})\ . \tag{3.36}$$

Confirm this condition is satisfied by

$$\sigma^{\alpha\beta} = \frac{i}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\ . \tag{3.37}$$

Now, let us define the finite transformation parameter $\Omega_{\mu\nu}$ by the limit

$$\lim_{N\to\infty} N\epsilon_{\mu\nu} = \Omega_{\mu\nu}\ . \tag{3.38}$$

The finite transformation matrix S is obtained by multiplying the infinitesimal matrix N times and taking the limit $N\to\infty$;

$$S = \lim_{N\to\infty}\left(1 - \frac{i}{4}\sigma^{\alpha\beta}\Omega_{\alpha\beta}/N\right)^N = \exp -\frac{i}{4}\sigma^{\alpha\beta}\Omega_{\alpha\beta}\ . \tag{3.39}$$

As for $\Lambda^\mu{}_\nu$, we define the 4×4 matrix $M^{\alpha\beta} = -M^{\beta\alpha}$ by the matrix element

$$\left(M^{\alpha\beta}\right)^\mu{}_\nu = i\left(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu\right)\ , \tag{3.40}$$

which realizes the infinitesimal transformation (3.33) by

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu - \frac{i}{2}\left(M^{\alpha\beta}\right)^\mu{}_\nu\epsilon_{\alpha\beta} = \delta^\mu_\nu + g^{\alpha\mu}\delta^\beta_\nu\epsilon_{\alpha\beta} = \delta^\mu_\nu + \epsilon^\mu{}_\nu\ . \tag{3.41}$$

The finite transformation $\Lambda^\mu{}_\nu$ is obtained by

$$\Lambda^\mu{}_\nu = \lim_{N\to\infty}\left(\left(1 - \frac{i}{2}M^{\alpha\beta}\Omega_{\alpha\beta}/N\right)^N\right)^\mu{}_\nu = \left(\exp -\frac{i}{2}M^{\alpha\beta}\Omega_{\alpha\beta}\right)^\mu{}_\nu\ . \tag{3.42}$$

Problem (3.2-2) Let us investigate the form of $\Lambda^\mu{}_\nu$ and S for the three-dimensional space rotation parametrized by Ω_{ij} $(i,j = 1,2,3)$. For this purpose, we introduce the angle θ, unit vector $\mathbf{n}$ $(|\mathbf{n}| = 1)$ and 4×4 matrix-valued vectors $\mathbf{M}$ and $\mathbf{S}$ by

$$\theta\, n^i = -\Omega_{jk}\ ,\quad M^i = M^{jk}\ ,\quad S^i = \frac{1}{2}\,\sigma^{jk}\ :\qquad (i,j,k) = \begin{cases} (1,2,3) \\ (2,3,1) \\ (3,1,2) \end{cases}\ . \tag{3.43}$$

Confirm the expressions

$$\Lambda^i{}_j = \left(\exp i\mathbf{M}\cdot\mathbf{n}\,\theta\right)^i{}_j\ , \qquad S = \exp i\mathbf{S}\cdot\mathbf{n}\,\theta\ . \tag{3.44}$$

Prove that **M** and **S** satisfy the algebra of the angular momentum

$$[M^i, M^j] = iM^k\ , \quad [S^i, S^j] = iS^k\ : \qquad (i,j,k) = \begin{cases} (1,2,3) \\ (2,3,1) \\ (3,1,2) \end{cases}\ . \tag{3.45}$$

Therefore, **S** must be identified as the spin matrix of the particle which the field ψ represents. Verify that **S** satisfies

$$\mathbf{S}^2 = s(s+1) \quad \text{with} \quad s = \frac{1}{2}\ . \tag{3.46}$$

This result definitely shows that the particle has spin $s = 1/2$. Taking $\mathbf{n} = (0,0,1)$, derive the expression

$$\Lambda^i{}_j = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\ . \tag{3.47}$$

This is the matrix of the coordinate rotation $\mathbf{x} \to \mathbf{x}' = \Lambda\mathbf{x}$ around x^3-axis by the angle θ. The general form of (3.44) represents the rotation around the axis specified by **n**. Confirm $S\psi$ does not return to its original value under the 2π rotation, but $S\psi = -\psi$ for $\theta = 2\pi$.

3.3 Free Dirac Field

The Lorentz invariant Lagrangian of the free Dirac field is given by

$$\mathcal{L} = \bar\psi(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\ . \tag{3.48}$$

The Euler-Lagrange equations are

$$0 = \frac{\partial\mathcal{L}}{\partial\bar\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)}\right) = (i\gamma^\mu\partial_\mu - m)\psi\ , \tag{3.49}$$

$$0 = \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = -\bar\psi m - \partial_\mu(\bar\psi i\gamma^\mu) = \bar\psi(-i\gamma^\mu\overleftarrow{\partial}_\mu - m)\ , \tag{3.50}$$

which coinside with the Dirac equations (3.11), (3.32).

Canonical Formalism : The canonical momentum conjugate to ψ is

$$\pi_{\psi_j} = \frac{\partial\mathcal{L}}{\partial\dot\psi_j} = (\bar\psi i\gamma^0)_j = i\psi_j^\dagger\ . \tag{3.51}$$

Since $\mathcal{L}$ does not contain $\dot{\bar\psi}$, the corresponding momentum $\pi_{\bar\psi}$ vanishes. Thus, the Hamiltonian density is

$$\begin{aligned} \mathcal{H}(\mathbf{x}) &= \pi_{\psi_j}\dot\psi_j - \mathcal{L} = \bar\psi i\gamma^0\dot\psi - (\bar\psi i\gamma^0\dot\psi + \bar\psi i\vec\gamma\cdot\nabla\psi - m\bar\psi\psi) \\ &= \psi^\dagger(\boldsymbol{\alpha}\cdot(-i\nabla) + m\beta)\psi\ , \end{aligned} \tag{3.52}$$

where

$$\alpha^i \equiv \gamma^0\gamma^i \ \ (i=1,2,3)\ , \ \ \beta \equiv \gamma^0 \quad ;\ \alpha^{i\dagger} = \alpha^i\ , \ \ \beta^\dagger = \beta\ . \tag{3.53}$$

Integrating $\mathcal{H}(\mathbf{x})$ over $\mathbf{x}$, we obtain the Hamiltonian

$$H = \int d^3\mathbf{x}\ \mathcal{H}(\mathbf{x}) = \int d^3\mathbf{x}\ \psi^\dagger(\boldsymbol{\alpha}\cdot(-i\nabla) + m\beta)\psi\ . \tag{3.54}$$

Quantization : From the prescription (3.8), the canonical anticommutators are

$$\{\hat{\psi}_j(\mathbf{x}), \hat{\pi}_{\psi_k}(\mathbf{y})\} = i\delta_{jk}\delta^3(\mathbf{x}-\mathbf{y})\ ,\quad \{\hat{\psi}_j(\mathbf{x}), \hat{\psi}_k(\mathbf{y})\} = 0\ ,\quad \{\hat{\pi}_{\psi_j}(\mathbf{x}), \hat{\pi}_{\pi_k}(\mathbf{y})\} = 0\ . \tag{3.55}$$

Since $\hat{\pi}_{\psi_k}(\mathbf{y}) = i\hat{\psi}_k^\dagger(\mathbf{y})$, we obtain

$$\{\hat{\psi}_j(\mathbf{x}), \hat{\psi}_k^\dagger(\mathbf{y})\} = \delta_{jk}\delta(\mathbf{x}-\mathbf{y})\ ,\quad \{\hat{\psi}_j(\mathbf{x}), \hat{\psi}_k(\mathbf{y})\} = 0\ ,\quad \{\hat{\psi}_j^\dagger(\mathbf{x}), \hat{\psi}_k^\dagger(\mathbf{y})\} = 0\ . \tag{3.56}$$

Heisenberg Picture : The field operators in Heisenberg picture satisfy the equal-time anticommutation relations

$$\{\hat{\psi}_j(\mathbf{x},t), \hat{\psi}_k^\dagger(\mathbf{y},t)\} = \delta_{jk}\delta(\mathbf{x}-\mathbf{y})\ ,\quad \{\hat{\psi}_j(\mathbf{x},t), \hat{\psi}_k(\mathbf{y},t)\} = 0 \quad : \hat{\psi}_j(\mathbf{x},t) = e^{i\hat{H}t}\hat{\psi}_j(\mathbf{x})e^{-i\hat{H}t}\ . \tag{3.57}$$

Since $\hat{H}(t) = \hat{H}$, Heisenberg's equation of motion for $\hat{\psi}$ is

$$\begin{aligned} i\ \frac{\partial}{\partial t}\hat{\psi}_j(\mathbf{x},t) &= [\hat{\psi}_j(\mathbf{x},t), \hat{H}(t)] = \int d^3\mathbf{y}[\hat{\psi}_j(\mathbf{x},t), \hat{\mathcal{H}}(\mathbf{y},t)] \\ &= \int d^3\mathbf{y}\ [\hat{\psi}_j(\mathbf{x},t), \hat{\psi}^\dagger(\mathbf{y},t)\left(\boldsymbol{\alpha}\cdot(-i\nabla_{(y)}) + m\beta\right)\hat{\psi}(\mathbf{y},t)] \\ &= \int d^3\mathbf{y}\ \{\hat{\psi}_j(\mathbf{x},t), \hat{\psi}^\dagger(\mathbf{y},t)\}\left(\boldsymbol{\alpha}\cdot(-i\nabla_{(y)}) + m\beta\right)\hat{\psi}(\mathbf{y},t) \\ &= \Big((\boldsymbol{\alpha}\cdot(-i\nabla) + m\beta)\hat{\psi}(\mathbf{x},t)\Big)_j\ , \end{aligned} \tag{3.58}$$

where, in the third line, we have used the anticommutation $\hat{\psi}_j(\mathbf{y},t)\hat{\psi}_k(\mathbf{x},t) = -\hat{\psi}_k(\mathbf{x},t)\hat{\psi}_j(\mathbf{y},t)$. This equation is equivalent to the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\hat{\psi}(\mathbf{x},t) = 0\ . \tag{3.59}$$

In the same way, we obtain the Dirac equation for $\hat{\bar{\psi}}$;

$$\hat{\bar{\psi}}(\mathbf{x},t)(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0\ . \tag{3.60}$$

Momentun Operator : Momentum operator $\hat{\mathbf{P}}$ is the operator which realizes

$$[\hat{\mathbf{P}}, \hat{\psi}(\mathbf{x},t)] = i\nabla\hat{\psi}(\mathbf{x},t) \tag{3.61}$$

$$[\hat{\mathbf{P}}, \hat{\psi}^\dagger(\mathbf{x},t)] = i\nabla\hat{\psi}^\dagger(\mathbf{x},t)\ . \tag{3.62}$$

According to $[\hat{\mathbf{P}}, \hat{H}] = 0$, $\hat{\mathbf{P}}$ is a conserved operator, $\hat{\mathbf{P}}(t) = \hat{\mathbf{P}}$. It is given by

$$\hat{\mathbf{P}}(t) = \int d^3\mathbf{x}\ \hat{\psi}^\dagger(\mathbf{x},t)(-i\nabla)\hat{\psi}(\mathbf{x},t)\ . \tag{3.63}$$

For example, we confirm

$$\begin{aligned} [\hat{\mathbf{P}}, \hat{\psi}(\mathbf{x},t)] &= \int d^3\mathbf{y}\ [\hat{\psi}^\dagger(\mathbf{y},t)(-i\nabla_{(y)})\hat{\psi}(\mathbf{y},t), \hat{\psi}(\mathbf{x},t)] \\ &= -\int d^3\mathbf{y}\ \{\hat{\psi}^\dagger(\mathbf{y},t), \hat{\psi}(\mathbf{x},t)\}(-i\nabla_{(y)})\hat{\psi}(\mathbf{y},t) \\ &= i\nabla\hat{\psi}(\mathbf{x},t)\ . \end{aligned} \tag{3.64}$$

3.4 Mode Expansion

Solutions of Dirac Equation : Since the solution of Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ satisfies the Klein-Gordon equation $(\Box + m^2)\psi(x) = 0$, it admits the plane-wave solutions

$$\psi(x) \sim u_{\mathbf{k}}\, e^{-ik\cdot x} , \quad v_{\mathbf{k}}\, e^{ik\cdot x} . \tag{3.65}$$

The 4-momentum k^μ satisfies the Einstein's relation

$$k^2 \equiv (k^0)^2 - \mathbf{k}^2 = m^2 , \qquad k^0 \equiv \omega_k = \sqrt{\mathbf{k}^2 + m^2} . \tag{3.66}$$

$u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are four component spinors which depend on k^μ, but not on x^μ. From Dirac equation, we find that $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ satisfy

$$(\not{k} - m)u_{\mathbf{k}} = 0 , \qquad (-\not{k} - m)v_{\mathbf{k}} = 0 , \tag{3.67}$$

where we have used the convenient notation invented by Feymman ;

$$\not{k} \equiv \gamma^\mu k_\mu , \qquad (\not{k})^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}k_\mu k_\nu = k^2 . \tag{3.68}$$

Taking the hermitian conjugete of equations (3.67) and multiplying them by γ^0 from the right, we obtain

$$\bar{u}_{\mathbf{k}}(\not{k} - m) = 0 , \qquad \bar{v}_{\mathbf{k}}(-\not{k} - m) = 0 . \tag{3.69}$$

Equations (3.67) and (3.69) are also called Dirac equations. Let us first investigate $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ with specific momentum $k^{(0)}_\mu = m\delta^0_\mu$, that is, $\mathbf{k} = \mathbf{0}$. In this case, (3.67) reduce to the simple form

$$m(\gamma^0 - 1)u_{\mathbf{0}} = 0 , \quad -m(\gamma^0 + 1)v_{\mathbf{0}} = 0 . \tag{3.70}$$

In the standard representation of γ^μ given by (3.15), we have

$$-\frac{1}{2}(\gamma^0 - 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \frac{1}{2}(\gamma^0 + 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.71}$$

Thus, the general solutions of (3.70) are

$$u_{\mathbf{0}} = \begin{pmatrix} \xi^1 \\ \xi^2 \\ 0 \\ 0 \end{pmatrix} , \qquad v_{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \\ \eta^1 \\ \eta^2 \end{pmatrix} . \tag{3.72}$$

This means that the degrees of freedom of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are two. This shows that the Dirac field $\psi(x)$ describes a particle with spin 1/2, whose spin degrees of freedom are just two.

The solutions with $\mathbf{k} \neq 0$ are obtained by the Lorentz transformation. Let $\Lambda_\mu{}^\nu$ be the transformation matrix which realizes

$$k_\mu = \Lambda_\mu{}^\nu k^{(0)}_\nu = m\Lambda_\mu{}^0 \tag{3.73}$$

and S be the corresponding transformation matrix for Lorentz spinor. Substituting γ^0 in (3.70) by the $\nu = 0$ component of (3.21), we have

$$0 = m(S^{-1}\gamma^\mu S\Lambda_\mu{}^0 - 1)u_{\mathbf{0}} = S^{-1}(\not{k} - m)Su_{\mathbf{0}} , \quad 0 = S^{-1}(-\not{k} - m)Sv_{\mathbf{0}} . \tag{3.74}$$

Thus, we find the solutions $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ of (3.67) are given by

$$u_{\mathbf{k}} = S u_{\mathbf{0}} \ , \qquad v_{\mathbf{k}} = S v_{\mathbf{0}} \ . \tag{3.75}$$

Owing to the Lorentz invariance of the bi-linears (3.30), $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ satisfy

$$\bar{u}_{\mathbf{k}} u_{\mathbf{k}} = \bar{u}_{\mathbf{0}} u_{\mathbf{0}} = u_{\mathbf{0}}^\dagger \gamma^0 u_{\mathbf{0}} = u_{\mathbf{0}}^\dagger u_{\mathbf{0}} > 0 \ , \tag{3.76}$$

$$\bar{v}_{\mathbf{k}} v_{\mathbf{k}} = \bar{v}_{\mathbf{0}} v_{\mathbf{0}} = v_{\mathbf{0}}^\dagger \gamma^0 v_{\mathbf{0}} = -v_{\mathbf{0}}^\dagger v_{\mathbf{0}} < 0 \ , \tag{3.77}$$

$$\bar{u}_{\mathbf{k}} v_{\mathbf{k}} = \bar{u}_{\mathbf{0}} v_{\mathbf{0}} = u_{\mathbf{0}}^\dagger \gamma^0 v_{\mathbf{0}} = -u_{\mathbf{0}}^\dagger v_{\mathbf{0}} = 0 \ , \tag{3.78}$$

$$\bar{v}_{\mathbf{k}} u_{\mathbf{k}} = 0 \ . \tag{3.79}$$

The orthogonality relation can be expressed in the other form. Multiplying the Dirac equations (3.67) by γ^0 from the left and using the anticommutatoion $\gamma^0\gamma^i = -\gamma^i\gamma^0$ $(i = 1, 2, 3)$, we observe that $\gamma^0 u_{\mathbf{k}}$ and $\gamma^0 v_{\mathbf{k}}$ satisfy the Dirac equations with momentum $-\mathbf{k}$, that is, $\gamma^0 u_{\mathbf{k}} \propto u_{-\mathbf{k}}$ and $\gamma^0 v_{\mathbf{k}} \propto v_{-\mathbf{k}}$. Therefore, we realize

$$u_{\mathbf{k}}^\dagger v_{-\mathbf{k}} = v_{\mathbf{k}}^\dagger u_{-\mathbf{k}} = 0 \ . \tag{3.80}$$

Basis Spinors : Let us define the basis spinors

$$u_{\mathbf{k},s} \ , \qquad v_{\mathbf{k},s} \ , \qquad s = 1, 2 \ , \tag{3.81}$$

of the solutions of the Dirac equations so that they satisfy the ortho-normality conditions

$$\bar{u}_{\mathbf{k},s} u_{\mathbf{k},s'} = 2m\delta_{ss'} \ , \quad \bar{v}_{\mathbf{k},s} v_{\mathbf{k},s'} = -2m\delta_{ss'} \ , \quad \bar{u}_{\mathbf{k},s} v_{\mathbf{k},s'} = \bar{v}_{\mathbf{k},s} u_{\mathbf{k},s'} = 0 \ . \tag{3.82}$$

The subscript s $(= 1, 2)$ represents the two possible spin states of the Dirac particle. As we can easily confirm, they generate the following "semi-completeness" matrices ;

$$\sum_{s=1,2} u_{\mathbf{k},s} \bar{u}_{\mathbf{k},s} \ = \ \not{k} + m \ , \tag{3.83}$$

$$\sum_{s=1,2} v_{\mathbf{k},s} \bar{v}_{\mathbf{k},s} \ = \ \not{k} - m \ . \tag{3.84}$$

For example, when (3.83) is multiplied to $v_{\mathbf{k},s'}$, both sides trivially vanish due to (3.82) and (3.67). When it is multiplied to $u_{\mathbf{k},s'}$, both sides turn out to $2m u_{\mathbf{k},s'}$.

From the Dirac equations (3.67) and (3.69), we obtain

$$\begin{aligned} 0 \ &= \ \bar{u}_{\mathbf{k},s} \left(\gamma^\mu (\not{k} - m) + (\not{k} - m)\gamma^\mu \right) u_{\mathbf{k},s'} \\ &= \ \bar{u}_{\mathbf{k},s} \left(\{\gamma^\mu, \gamma^\nu\} k_\nu - 2m\gamma^\mu \right) u_{\mathbf{k},s'} = \bar{u}_{\mathbf{k},s} \left(2k^\mu - 2m\gamma^\mu \right) u_{\mathbf{k},s'} \ , \end{aligned} \tag{3.85}$$

and similarly, $\bar{v}_{\mathbf{k},s} \left(2k^\mu + 2m\gamma^\mu \right) v_{\mathbf{k},s'} = 0$. The normalization conditions (3.82) then give

$$\bar{u}_{\mathbf{k},s} \gamma^\mu u_{\mathbf{k},s'} = 2k^\mu \delta_{ss'} \ , \qquad \bar{v}_{\mathbf{k},s} \gamma^\mu v_{\mathbf{k},s'} = 2k^\mu \delta_{ss'} \ . \tag{3.86}$$

Mode Expansion : Now, we are ready to expand $\hat{\psi}(\mathbf{x}, t)$ in terms of the mode functions ;

$$\hat{\psi}(\mathbf{x}, t) = \sum_{s=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{b}_{\mathbf{k},s} u_{\mathbf{k},s} e^{-ik\cdot x} + \hat{d}_{\mathbf{k},s}^\dagger v_{\mathbf{k},s} e^{ik\cdot x} \right) \ . \tag{3.87}$$

The anticommutation relations $\{\hat{\psi}_j(\mathbf{x}, t) \hat{\psi}_k^\dagger(\mathbf{y}, t)\} = \delta_{jk}\delta^3(\mathbf{x} - \mathbf{y})$, etc. determine the anticommutation relations of $\hat{b}_{\mathbf{k},s}$ and $\hat{d}_{\mathbf{k},s}$. The result is

$$\{\hat{b}_{\mathbf{k},s}, \hat{b}_{\mathbf{k}',s'}^\dagger\} = \{\hat{d}_{\mathbf{k},s}, \hat{d}_{\mathbf{k}',s'}^\dagger\} = \delta_{ss'}\delta^3(\mathbf{k} - \mathbf{k}') \ , \quad \text{other } \{*, *\} = 0 \ . \tag{3.88}$$

For example, we can confirm

$$
\begin{aligned}
\{\hat{\psi}_j(\mathbf{x},t)\hat{\psi}_k^\dagger(\mathbf{y},t)\} &= \int \frac{d^3\mathbf{k}d^3\mathbf{k}'}{(2\pi)^3}\sum_{s,s'}\frac{1}{2\sqrt{\omega_k\omega_{k'}}}\Big(\{\hat{b}_{\mathbf{k},s},\hat{b}^\dagger_{\mathbf{k}',s'}\}u_{\mathbf{k},s}u^\dagger_{\mathbf{k}',s'}e^{-i(k\cdot x-k'\cdot y)} \qquad \left\{\begin{array}{l} x=(t,\mathbf{x}) \\ y=(t,\mathbf{y})\end{array}\right. \\
&\qquad\qquad +\{\hat{d}_{\mathbf{k},s},\hat{d}^\dagger_{\mathbf{k}',s'}\}v_{\mathbf{k},s}v^\dagger_{\mathbf{k}',s'}e^{i(k\cdot x-k'\cdot y)}\Big)_{jk} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\sum_s\frac{1}{2\omega_k}\Big(u_{\mathbf{k},s}u^\dagger_{\mathbf{k},s}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+v_{\mathbf{k},s}v^\dagger_{\mathbf{k},s}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\Big)_{jk} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_k}\big((\not{k}+m)\gamma^0e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+(\not{k}-m)\gamma^0e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\big)_{jk} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}(1)_{jk} \\
&= \delta_{jk}\delta^3(\mathbf{x}-\mathbf{y}) \ ,
\end{aligned} \tag{3.89}
$$

where, in the second term of the third line, we have changed the integration variable $\mathbf{k}$ to $-\mathbf{k}$.

Hamiltonian : The quantum Hamiltonian is obtained by replacing ψ and $\psi^\dagger$ in the classical Hamiltonian (3.54) with $\hat{\psi}$ and $\hat{\psi}^\dagger$. From (3.67), we find $u_{\mathbf{k},s}$ and $v_{\mathbf{k},s}$ satisfy

$$
(\mathbf{k}\cdot\boldsymbol{\alpha}+m\beta)u_{\mathbf{k},s}=\omega_k u_{\mathbf{k},s} \ , \quad (-\mathbf{k}\cdot\boldsymbol{\alpha}+m\beta)v_{\mathbf{k},s}=-\omega_k v_{\mathbf{k},s} \qquad : \ \alpha^i=\gamma^0\gamma^i \ , \ \ \beta=\gamma^0 \ . \tag{3.90}
$$

Therefore, we have

$$
(\boldsymbol{\alpha}\cdot(-i\nabla)+m\beta)\,\hat{\psi}(\mathbf{x},t)=\sum_{s=1,2}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\Big(\hat{b}_{\mathbf{k},s}u_{\mathbf{k},s}e^{-ik\cdot x}-\hat{d}^\dagger_{\mathbf{k},s}v_{\mathbf{k},s}e^{ik\cdot x}\Big) \ . \tag{3.91}
$$

Incorporating this expression to the Hamiltonian $\hat{H}(t)$, we obtain

$$
\begin{aligned}
\hat{H}(t) &= \int d^3\mathbf{x}\ \hat{\psi}^\dagger(\mathbf{x},t)\,(\boldsymbol{\alpha}\cdot(-i\nabla)+m\beta)\,\hat{\psi}(\mathbf{x},t) \\
&= \int d^3\mathbf{x}\sum_{s'=1,2}\int\frac{d^3\mathbf{k}'}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{k'}}}\Big(\hat{b}^\dagger_{\mathbf{k}',s'}u^\dagger_{\mathbf{k}',s'}e^{ik'\cdot x}+\hat{d}_{\mathbf{k}',s'}v^\dagger_{\mathbf{k}',s'}e^{-ik'\cdot x}\Big) \\
&\qquad\qquad \times\sum_{s=1,2}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\Big(\hat{b}_{\mathbf{k},s}u_{\mathbf{k},s}e^{-ik\cdot x}-\hat{d}^\dagger_{\mathbf{k},s}v_{\mathbf{k},s}e^{ik\cdot x}\Big) \\
&= \sum_{s,s'}\int d^3\mathbf{k}'d^3\mathbf{k}\ \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\int\frac{d^3\mathbf{x}}{(2\pi)^3}\Big(\hat{b}^\dagger_{\mathbf{k}',s'}\hat{b}_{\mathbf{k},s}u^\dagger_{\mathbf{k}',s'}u_{\mathbf{k},s}e^{i(k'-k)\cdot x}-\hat{d}_{\mathbf{k}',s'}\hat{d}^\dagger_{\mathbf{k},s}v^\dagger_{\mathbf{k}',s'}v_{\mathbf{k},s}e^{i(-k'+k)\cdot x} \\
&\qquad\qquad -\hat{b}^\dagger_{\mathbf{k}',s'}\hat{d}^\dagger_{\mathbf{k},s}u^\dagger_{\mathbf{k}',s'}v_{\mathbf{k},s}e^{i(k'+k)\cdot x}+\hat{d}_{\mathbf{k}',s'}\hat{b}_{\mathbf{k},s}v^\dagger_{\mathbf{k}',s'}u_{\mathbf{k},s}e^{-i(k'+k)\cdot x}\Big) \\
&= \sum_{s,s'}\int d^3\mathbf{k}\ \frac{1}{2}\Big(\hat{b}^\dagger_{\mathbf{k},s'}\hat{b}_{\mathbf{k},s}u^\dagger_{\mathbf{k},s'}u_{\mathbf{k},s}-\hat{d}_{\mathbf{k},s'}\hat{d}^\dagger_{\mathbf{k},s}v^\dagger_{\mathbf{k},s'}v_{\mathbf{k},s} \\
&\qquad\qquad -\hat{b}^\dagger_{-\mathbf{k},s'}\hat{d}^\dagger_{\mathbf{k},s}u^\dagger_{-\mathbf{k},s'}v_{\mathbf{k},s}e^{i2\omega_kt}+\hat{d}_{-\mathbf{k},s'}\hat{b}_{\mathbf{k},s}v^\dagger_{-\mathbf{k},s'}u_{\mathbf{k},s}e^{-i2\omega_kt}\Big) \\
&= \sum_s\int d^3\mathbf{k}\ \omega_k(\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s}-\hat{d}_{\mathbf{k},s}\hat{d}^\dagger_{\mathbf{k},s})
\end{aligned} \tag{3.92}
$$

where, in the fourth line, we have used the $\mu = 0$ components of (3.86) and the orthogonality (3.80). Notice that the time dependence disappeared from $\hat{H}(t)$ as it should be. Owing to the anticommutators (3.88), Hamiltonian is expressed as

$$\hat{H} = \sum_{s=1,2} \int d^3\mathbf{k}\ \omega_k(\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} + \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}) - [\text{const.}]\ . \tag{3.93}$$

Although we omit the constant term [const.], it is instructive to notice that the Dirac field gives the negative energy density $\epsilon_{\text{vac}} = -\int d^3\mathbf{k}\ \omega_k/(4\pi^3)$ to the space, four-times larger in the magnitude than that of real scalar field. The factor four comes from the fact that the complex field $\hat{\psi}$ describes particle and antiparticle, both having two spin degrees of freedom.

As in the case of complex scalar field,

$$\left\{ \begin{array}{c} \hat{b}^\dagger, \hat{b} \\ \hat{d}^\dagger, \hat{d} \end{array} \right\} \text{create/annihirate} \left\{ \begin{array}{c} \text{particle} \\ \text{antiparticle} \end{array} \right\} .$$

The vacuum $|0\rangle$ is defined, for all annihilation operators $\hat{b}_{\mathbf{k},s}$ and $\hat{d}_{\mathbf{k},s}$, by

$$\hat{b}_{\mathbf{k},s}|0\rangle = 0\ ,\quad \hat{d}_{\mathbf{k},s}|0\rangle = 0\ . \tag{3.94}$$

Remark : If $\hat{\psi}$ were quantized by the Bose-Einstein statistics, Hamiltonian would take a form

$$\hat{H} = \sum_{s=1,2} \int d^3\mathbf{k}\ \omega_k(\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} - \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}) - [\text{constant}]\ .$$

This implies that the antiparticle appears with negative energy $-\omega_k$, so the creation of antiparticle decreases the energy of the state and we cannot define the vacuum as the stable lowest energy state. Thus, the spin 1/2 particle must be subject to the Fermi-Dirac statistics.

Momentum Operator : With the similar procedure taken for the calculation of $\hat{H}$, we arrive at

$$\begin{aligned} \hat{\mathbf{P}}(t) &= \hat{\mathbf{P}} = \int d^3\mathbf{x}\ \hat{\psi}^\dagger(\mathbf{x},t)(-i\nabla)\hat{\psi}(\mathbf{x},t) \\ &= \sum_{s=1,2} \int d^3\mathbf{k}\ \mathbf{k}(\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} - \hat{d}_{\mathbf{k},s}\hat{d}^\dagger_{\mathbf{k},s}) \\ &= \sum_{s=1,2} \int d^3\mathbf{k}\ \mathbf{k}(\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} + \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s})\ . \end{aligned} \tag{3.95}$$

In the last line, the constant term due to the anticommutator $\{\hat{d}_{\mathbf{k},s}, \hat{d}^\dagger_{\mathbf{k},s}\}$ vanishes after $\mathbf{k}$ integration owing to the odd factor $\mathbf{k}$.

Charge : From the Dirac equation we obtain the following equations :

$$\begin{array}{rl} & \hat{\bar{\psi}} \times\ (i\gamma^\mu\partial_\mu - m)\hat{\psi} = 0 \\ -) & \bar{\psi}(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0\ \times \hat{\psi} \\ \hline & i\partial_\mu(\hat{\bar{\psi}}\gamma^\mu\hat{\psi}) = 0 \end{array} \tag{3.96}$$

Thus, the current defined by

$$\hat{j}^\mu(\mathbf{x},t) = \hat{\bar{\psi}}(\mathbf{x},t)\gamma^\mu\hat{\psi}(\mathbf{x},t)\ , \tag{3.97}$$

satisfies the continuity equation

$$\partial_\mu \hat{j}^\mu(\mathbf{x},t) = 0 \ . \tag{3.98}$$

The space integral of $\mu = 0$ component of $\hat{j}^\mu$ gives the conserved charge

$$\begin{aligned}\hat{Q}(t) = \hat{Q} &= \int d^3\mathbf{x}\hat{j}^0(\mathbf{x},t) = \int d^3\mathbf{x}\ \hat{\psi}^\dagger(\mathbf{x},t)\hat{\psi}(\mathbf{x},t) \\ &= \sum_{s=1,2}\int d^3\mathbf{k}\ (\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} + \hat{d}_{\mathbf{k},s}\hat{d}^\dagger_{\mathbf{k},s}) \\ &= \sum_{s=1,2}\int d^3\mathbf{k}\ (\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} - \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}) + [\text{constant}] \ . \end{aligned} \tag{3.99}$$

The [constant] due to the anti-commutator $\{\hat{d}_{\mathbf{k},s}, \hat{d}_{\mathbf{k},s}\}$ is an artifact of the definition of $\hat{j}^\mu(x)$. If we define $\hat{j}^\mu$ "symmetrically" for particle and antiparticle, we will obtain

$$\hat{Q} = \frac{1}{2}\sum_{s=1,2}\int d^3\mathbf{k}(\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} + \hat{d}_{\mathbf{k},s}\hat{d}^\dagger_{\mathbf{k},s} - \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s} - \hat{b}_{\mathbf{k},s}\hat{b}^\dagger_{\mathbf{k},s}) = \sum_{s=1,2}\int d^3\mathbf{k}\ (\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} - \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}) \ . \tag{3.100}$$

Therefore, the charge of vacuum vanishes, $\hat{Q}|0\rangle = 0$. From the commutators

$$[\hat{Q}, \hat{b}^\dagger_{\mathbf{k},s}] = b^\dagger_{\mathbf{k},s} \ , \qquad [\hat{Q}, \hat{d}^\dagger_{\mathbf{k},s}] = -d^\dagger_{\mathbf{k},s} \ , \tag{3.101}$$

we see that the particle has charge +1, and the antiparticle has −1 :

$$\hat{Q}\hat{b}^\dagger_{\mathbf{k},s}|0\rangle = ([\hat{Q}, \hat{b}^\dagger_{\mathbf{k},s}] + \hat{b}^\dagger_{\mathbf{k},s}\hat{Q})|0\rangle = +\hat{b}^\dagger_{\mathbf{k},s}|0\rangle \ , \tag{3.102}$$

$$\hat{Q}\hat{d}^\dagger_{\mathbf{k},s}|0\rangle = ([\hat{Q}, \hat{d}^\dagger_{\mathbf{k},s}] + \hat{d}^\dagger_{\mathbf{k},s}\hat{Q})|0\rangle = -\hat{d}^\dagger_{\mathbf{k},s}|0\rangle \ . \tag{3.103}$$

Problem (3.4-1) In the standard representation (3.15) of the Dirac's matrix γ^μ, the basis spinors are given by

$$u_{\mathbf{k},s} = \sqrt{\omega_k + m}\begin{pmatrix} \xi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_k+m}\xi_s \end{pmatrix} \ , \quad v_{\mathbf{k},s} = \sqrt{\omega_k + m}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_k+m}\eta_s \\ \eta_s \end{pmatrix} \ , \qquad s = 1,2 \ , \tag{3.104}$$

where ξ_s and η_s are two component spinors satisfying

$$\xi^\dagger_s\xi_{s'} = \delta_{ss'} \ , \quad \eta^\dagger_s\eta_{s'} = \delta_{ss'} \ , \qquad \text{for example} \ \ \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ spin} \uparrow \ , \ \ \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ spin} \downarrow \ . \tag{3.105}$$

Check that (3.104) satisfy the Dirac equations

$$(\not{k} - m)u_{\mathbf{k},s} = 0 \ , \qquad (-\not{k} - m)v_{\mathbf{k},s} = 0 \ , \tag{3.106}$$

orthonormality conditions

$$\bar{u}_{\mathbf{k},s}u_{\mathbf{k},s'} = 2m\delta_{ss'} \ , \quad \bar{v}_{\mathbf{k},s}v_{\mathbf{k},s'} = -2m\delta_{ss'} \ , \quad \bar{u}_{\mathbf{k},s}v_{\mathbf{k},s'} = \bar{v}_{\mathbf{k},s}u_{\mathbf{k},s'} = 0 \ , \tag{3.107}$$

and "semi-completeness" relations

$$\sum_{s=1,2} u_{\mathbf{k},s}\bar{u}_{\mathbf{k},s} = \not{k} + m \ , \qquad \sum_{s=1,2} v_{\mathbf{k},s}\bar{v}_{\mathbf{k},s} = \not{k} - m \ . \tag{3.108}$$

Problem (3.4-2) Confirm the equal-time anticommutations

$$\{\hat{\psi}_j(\mathbf{x},t), \hat{\psi}_k(\mathbf{y},t)\} = \{\hat{\psi}_j^\dagger(\mathbf{x},t), \hat{\psi}_k^\dagger(\mathbf{y},t)\} = 0 \tag{3.109}$$

of the free Dirac field $\hat{\psi}(\mathbf{x},t)$ and $\hat{\psi}^\dagger(\mathbf{x},t)$ by the anticommutators of creation and annihiration operators

$$\{\hat{b}_{\mathbf{k},s}, \hat{b}_{\mathbf{k}',s'}^\dagger\} = \{\hat{d}_{\mathbf{k},s}, \hat{d}_{\mathbf{k}',s'}^\dagger\} = \delta_{ss'}\delta^3(\mathbf{k}-\mathbf{k}') \ , \quad \text{other } \{*,*\} = 0 \ . \tag{3.110}$$

Problem (3.4-3) Derive the expression of the momentum operator (3.95)

$$\hat{\mathbf{P}} = \sum_{s=1,2} \int d^3\mathbf{k} \ \mathbf{k}(\hat{b}_{\mathbf{k},s}^\dagger \hat{b}_{\mathbf{k},s} - \hat{d}_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}^\dagger) \tag{3.111}$$

and the charge operator (3.99)

$$\hat{Q} = \sum_{s=1,2} \int d^3\mathbf{k} \ (\hat{b}_{\mathbf{k},s}^\dagger \hat{b}_{\mathbf{k},s} + \hat{d}_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}^\dagger) \tag{3.112}$$

by carrying out the space integration.

3.5 Charge Conjugation

Taking the transpose of the Dirac equation $\hat{\bar{\psi}}(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0$, we obtain

$$0 = \left(\hat{\bar{\psi}}(-i\gamma^\mu\overleftarrow{\partial}_\mu - m)\right)^T = \left(-i(\gamma^\mu)^T\partial_\mu - m\right)\hat{\bar{\psi}}^T \ . \tag{3.113}$$

According to the unitary equivalence of γ^μ-matrices, we always have the unitary matrix C which realizes

$$-(\gamma^\mu)^T = C^{-1}\gamma^\mu C \ , \qquad C^T = -C \ . \tag{3.114}$$

In the standard representation, it is given by $C = i\gamma^2\gamma^0$. Therefore, multiplying (3.113) by C from the left, we see that

$$\hat{\psi}^c \equiv C\hat{\bar{\psi}}^T \tag{3.115}$$

also satisfies the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\hat{\psi}^c = 0 \ . \tag{3.116}$$

$\hat{\psi}^c$ is called the charge conjugate field of $\hat{\psi}$, and $(\hat{\psi}^c)^c = \hat{\psi}$ is realized. The mode expansion of $\hat{\psi}^c$ is

$$\hat{\psi}^c(\mathbf{x},t) = \sum_{s=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{d}_{\mathbf{k},s} C\bar{v}_{\mathbf{k},s}^T e^{-ik\cdot x} + \hat{b}_{\mathbf{k},s}^\dagger C\bar{u}_{\mathbf{k},s}^T e^{ik\cdot x}\right) \ . \tag{3.117}$$

Thus, $\hat{\psi}^c$ annihirates antiparticle and creates particle.

If we relate the basis spinors $u_{\mathbf{k},s}$ and $v_{\mathbf{k},s}$ by

$$v_{\mathbf{k},s} = C\bar{u}_{\mathbf{k},s}^T \ , \quad u_{\mathbf{k},s} = C\bar{v}_{\mathbf{k},s}^T \ , \tag{3.118}$$

then, it gives the completely equivalent expression for particle and antiparticle. In the standard representation (3.104), this amounts to taking the two-component spinors ξ_s and η_s in a way $\eta_s = -i\sigma^2\xi_s^*$. It is perfectly admissible to describe the quantum system of Dirac field using $\hat{\psi}^c$ instead of $\hat{\psi}$. It is a matter of convention. Antiparticle of the antiparticle is particle.

Problem (3.5-1) Suppose the Dirac field $\psi(x)$ transforms under Lorentz transformation as $\psi(x) \to \psi'(x') = S\psi(x)$. Show the charge conjugate field $\psi^c(x)$ also transforms as $\psi^c(x) \to \psi^{c\prime}(x') = S\psi^c(x)$. That is, $\psi^c(x)$ is a Lorentz spinor.

Problem (3.5-2) Supose the Dirac field $\hat{\psi}(x)$ is interacting with the classical electro-magnetic field $A^\mu(x)$. The equation of motion is given by

$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu(x) - m)\hat{\psi}(x) = 0 \ , \tag{3.119}$$

where e is an electric charge of the particle $\hat{\psi}(x)$ represents. Rewrite this equation in terms of $\hat{\psi}^c(x)$ and derive the expression

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu(x) - m)\hat{\psi}^c(x) = 0 \ . \tag{3.120}$$

This result shows that the antiparticle has an electric charge $-e$.

3.6 T-Product and Propagator of Fermi Field

When the T-product of the field operators contains the fermi fields, T-product is defined to accommodate a minus sign associated with the interchange of fermi fields. For example,

$$\begin{aligned}
T[\hat{\psi}_j(x_1)\hat{\bar{\psi}}_k(x_2)] &= \begin{cases} \hat{\psi}_j(x_1)\hat{\bar{\psi}}_k(x_2) & x_1^0 > x_2^0 \\ -\hat{\bar{\psi}}_k(x_2)\hat{\psi}_j(x_1) & x_2^0 > x_1^0 \end{cases} \\
&= \theta(x_1^0 - x_2^0)\hat{\psi}_j(x_1)\hat{\bar{\psi}}_k(x_2) - \theta(x_2^0 - x_1^0)\hat{\bar{\psi}}_k(x_2)\hat{\psi}_j(x_1) \\
&= -T[\hat{\bar{\psi}}_k(x_2)\hat{\psi}_j(x_1)] \ .
\end{aligned} \tag{3.121}$$

Propagator of Free Dirac Field : Let us define the propagator of the free Dirac field by

$$\langle 0|T[\hat{\psi}_j(x_1)\hat{\bar{\psi}}_k(x_2)]|0\rangle = i\left(S_F(x_1 - x_2)\right)_{jk} \ . \tag{3.122}$$

As in the case of the complex scalar field, the charge conservation gives

$$\langle 0|T[\hat{\psi}_j(x_1)\hat{\psi}_k(x_2)]|0\rangle = \langle 0|T[\hat{\bar{\psi}}_j(x_1)\hat{\bar{\psi}}_k(x_2)]|0\rangle = 0 \ . \tag{3.123}$$

The manipulation used for the scalar field in II.4 leads to the expression for $S_F(x_1 - x_2)$ as

$$\begin{aligned}
S_F(x_1 - x_2) &= \int \frac{d^4k}{(2\pi)^4}\frac{\not{k} + m}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x_1 - x_2)} \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{1}{\not{k} - m + i\epsilon}e^{-ik\cdot(x_1 - x_2)} \ ,
\end{aligned} \tag{3.124}$$

where we have used the identity $(\not{k})^2 = k^2$. The Dirac propagator $S_F(x)$ satisfies

$$(i\not{\partial} - m)S_F(x) = \delta^4(x) \ . \tag{3.125}$$

It is related to the scalar propagator $\Delta_F(x)$ by

$$S_F(x) = (i\not{\partial} + m)\Delta_F(x) \ . \tag{3.126}$$

Problem (3.6-1) Derive the expression (3.124) for the propagator of the Dirac field $\hat{\psi}(x)$.

3.7 Chapter-Project/F

According to the modern understanding of the particle physics, the mass of the particle is not an inherent attribute of the particle but is a consequence of the dynamics each particle undergoes. Therefore, the mass m can no longer be a constant for the particle but is a variable quantity depending on the evolution of the circumstances surrounding the particle, which may be most eminently visualized in the early stage of the universe.

Suppose the mass $m(t)$ of the Dirac field ψ changes its value from m_0 to m in accordance with the change of its surroundings. We may expect something will happen owing to this change of mass. For the sake of simplicity of the analysis, we assume the change of the mass $m(t)$ is sudden at $t = 0$;

$$m(t) = m_0 \quad (t \leq 0) \ , \qquad m(t) = m \quad (t > 0) \ . \tag{3.127}$$

In the Schrödinger picture, the quantum field operator $\hat{\psi}(\mathbf{x})$ has no time dependence. We can expand $\hat{\psi}(\mathbf{x})$ in terms of the old ($t \leq 0$) mode functions $u^0_{\mathbf{k},s}$, $v^0_{\mathbf{k},s}$ evaluated with the mass m_0, or in terms of the new ($t > 0$) mode functions $u_{\mathbf{k},s}$, $v_{\mathbf{k},s}$ evaluated with the mass m :

$$\hat{\psi}(\mathbf{x}) = \sum_{s=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega^0_k}} \left(\hat{b}^0_{\mathbf{k},s} u^0_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{d}^{0\dagger}_{\mathbf{k},s} v^0_{\mathbf{k},s} e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \qquad \omega^0_k = \sqrt{\mathbf{k}^2 + m_0^2} \ , \tag{3.128}$$

$$\hat{\psi}(\mathbf{x}) = \sum_{s=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{b}_{\mathbf{k},s} u_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{d}^{\dagger}_{\mathbf{k},s} v_{\mathbf{k},s} e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \qquad \omega_k = \sqrt{\mathbf{k}^2 + m^2} \ . \tag{3.129}$$

Let us assume the state $|\psi(t)\rangle$ at $t \leq 0$ is the vacuum $|0\rangle$ of the old Hamiltonian H^0 and satisfies

$$\hat{b}^0_{\mathbf{k},s}|\psi(0)\rangle = 0 \ , \qquad \hat{d}^0_{\mathbf{k},s}|\psi(0)\rangle = 0 \ . \tag{3.130}$$

The particle picture at $t > 0$ is described by the new operators $\hat{b}$ and $\hat{d}$, which will no longer realize $\hat{b}|\psi(0)\rangle = 0$ nor $\hat{d}|\psi(0)\rangle = 0$. Therefore, we expect the state at $t > 0$ is not the vacuum but will contain number of particles. The particle number operator $\hat{N}$ is easily verified to be

$$\hat{N} = \sum_{s=1,2} \int d^3\mathbf{k} \ (\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} + \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}) \ . \tag{3.131}$$

Let us concentrate on the differential quantity

$$d\hat{N}_{\mathbf{k}} = d^3\mathbf{k} \sum_{s=1,2} (\hat{b}^\dagger_{\mathbf{k},s}\hat{b}_{\mathbf{k},s} + \hat{d}^\dagger_{\mathbf{k},s}\hat{d}_{\mathbf{k},s}) \ . \tag{3.132}$$

Since $d\hat{N}_{\mathbf{k}}$ commutes with the Hamiltonian $\hat{H}$ at $t > 0$, the differential particle number $\langle dN_{\mathbf{k}}(t)\rangle$ is expressed as

$$\langle dN_{\mathbf{k}}(t)\rangle \equiv \langle\psi(t)|d\hat{N}_{\mathbf{k}}|\psi(t)\rangle = \langle\psi(0)|d\hat{N}_{\mathbf{k}}|\psi(0)\rangle \ . \tag{3.133}$$

The completeness of the plane wave functions in (3.128) and (3.129) gives the operator relation

$$\sum_{s=1,2} \frac{1}{\sqrt{2\omega^0_k}} \left(\hat{b}^0_{\mathbf{k},s} u^0_{\mathbf{k},s} + \hat{d}^{0\dagger}_{-\mathbf{k},s} v^0_{-\mathbf{k},s} \right) = \sum_{s=1,2} \frac{1}{\sqrt{2\omega_k}} \left(\hat{b}_{\mathbf{k},s} u_{\mathbf{k},s} + \hat{d}^\dagger_{-\mathbf{k},s} v_{-\mathbf{k},s} \right) \ . \tag{3.134}$$

This relation enables us to express $\hat{b}$ and $\hat{d}^\dagger$ in terms of $\hat{b}^0$ and $\hat{d}^{0\dagger}$ in the form

$$\hat{b}_{\mathbf{k},s} = \sum_{s'=1,2} \left(\alpha_{ss'} \hat{b}^0_{\mathbf{k},s'} + \beta_{ss'} \hat{d}^{0\dagger}_{-\mathbf{k},s'} \right) \ , \quad \hat{d}^\dagger_{-\mathbf{k},s} = \sum_{s'=1,2} \left(\tilde{\alpha}_{ss'} \hat{d}^{0\dagger}_{-\mathbf{k},s'} + \tilde{\beta}_{ss'} \hat{b}^0_{\mathbf{k},s'} \right) \ , \tag{3.135}$$

which is a kind of Bogoliubov transformation. The coefficient matrices are determined using (3.80) and (3.86) as

$$\alpha_{ss'} = \frac{u^\dagger_{\mathbf{k},s} u^0_{\mathbf{k},s'}}{2\sqrt{\omega_k \omega^0_k}} \ , \quad \beta_{ss'} = \frac{u^\dagger_{\mathbf{k},s} v^0_{-\mathbf{k},s'}}{2\sqrt{\omega_k \omega^0_k}} \ , \quad \tilde{\alpha}_{ss'} = \frac{v^\dagger_{-\mathbf{k},s} v^0_{-\mathbf{k},s'}}{2\sqrt{\omega_k \omega^0_k}} \ , \quad \tilde{\beta}_{ss'} = \frac{v^\dagger_{-\mathbf{k},s} u^0_{\mathbf{k},s'}}{2\sqrt{\omega_k \omega^0_k}} \ . \tag{3.136}$$

Since the operators which survive in the matrix element (3.133) are $\hat{b}^0\hat{b}^{0\dagger}$ and $\hat{d}^0\hat{d}^{0\dagger}$, $\langle dN_{\mathbf{k}} \rangle$ is expressed as

$$\begin{aligned}
\langle dN_{\mathbf{k}} \rangle &= d^3\mathbf{k} \sum_{s=1,2} \langle \psi(0)| \sum_{s',s''} \left(\beta^*_{ss'}\beta_{ss''} \hat{d}^0_{-\mathbf{k},s'} \hat{d}^{0\dagger}_{-\mathbf{k},s''} + \tilde{\beta}_{ss'}\tilde{\beta}^*_{ss''} \hat{b}^0_{\mathbf{k},s'} \hat{b}^{0\dagger}_{\mathbf{k},s''} \right) |\psi(0)\rangle \\
&= d^3\mathbf{k}\ \delta^3(\mathbf{k}-\mathbf{k}) \sum_{s,s'} \left(\beta^*_{ss'}\beta_{ss'} + \tilde{\beta}_{ss'}\tilde{\beta}^*_{ss'} \right) \ .
\end{aligned} \tag{3.137}$$

The calculation of $\sum_{s,s'} \beta^*_{ss'}\beta_{ss'}$ can be performed using (3.83) and (3.84) as

$$\begin{aligned}
\sum_{s,s'=1,2} \beta^*_{ss'}\beta_{ss'} &= \sum_{s,s'=1,2} \frac{1}{4\omega_k\omega^0_k} v^{0\dagger}_{-\mathbf{k},s'} u_{\mathbf{k},s} u^\dagger_{\mathbf{k},s} v^0_{-\mathbf{k},s'} \\
&= \frac{1}{4\omega_k\omega^0_k} \mathrm{tr}(\not{k}+m)(\not{k}_0 - m_0) \qquad \text{where} \ \ k^\mu_0 = (\omega^0_k, \mathbf{k}) \\
&= \frac{1}{4\omega_k\omega^0_k} \mathrm{tr}(\not{k}\not{k}_0 - m m_0) \qquad \text{because} \ \ \mathrm{tr}(\gamma^\mu) = 0 \\
&= \frac{\omega_k\omega^0_k - k^2 - m m_0}{\omega_k\omega^0_k} \qquad \text{because} \ \ \mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \ .
\end{aligned} \tag{3.138}$$

The second term of (3.137), which represents the number of antiparticles, gives the same result which reflects the symmetry between particles and antiparticles. Noticing $\delta^3(\mathbf{k}-\mathbf{k}) = V/(2\pi)^3$, we arrive at the expression for the differential particle number density created by the change of mass :

$$dn_{\mathbf{k}} \equiv \frac{\langle dN_{\mathbf{k}} \rangle}{V} = d^3\mathbf{k}\ \frac{1}{4\pi^3} \frac{\omega_k\omega^0_k - k^2 - m m_0}{\omega_k\omega^0_k} \ . \tag{3.139}$$

We should notice that the total number density $n = \int dn_{\mathbf{k}}$ diverges in the region $|\mathbf{k}| \to \infty$. This divergence is due to the sudden change of the mass.

Chapter 4

Interaction of Fields

4.1 Interacting Fields

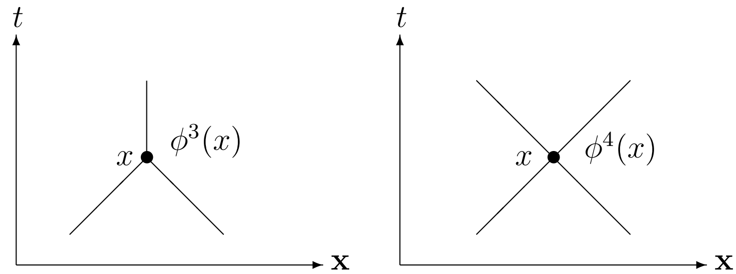
Free fields have been described by the Lagrangian $\mathcal{L}_0$ which consists of the quadratic terms of the field $\phi(x)$. Interacting fields are described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0[\text{quadratic}] + \mathcal{L}_{\text{int}}[\text{higher polynomial}] \ , \tag{4.1}$$

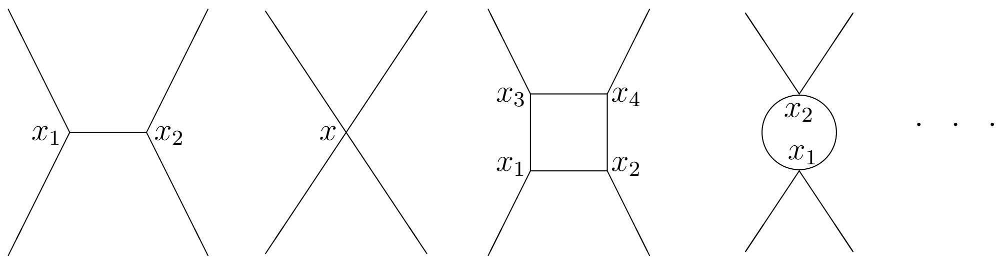
where $\mathcal{L}_{\text{int}}$ consists of cubic and higher terms of the fields ;

$$\mathcal{L}_{\text{int}} \sim \quad \phi^3(x) \ + \ \phi^4(x) \ + \ \cdots \ . \tag{4.2}$$

Since ϕ creates or annihirates one particle, the term $\phi^3(x)$ ($\phi^4(x)$) represents the process where three (four) particles, in total, participate the interaction at the space-time point x. In a graphical representation, they are expressed as follows ;



Distributing these interactions over various space-time points x_i and connecting the interaction points by the propagators, we obtain the rich variety of patterns of the interaction processes :



The main subject of the quantum field theory is to derive the quantum-mechanical amplitudes which describe these processes. This is a quite nontrivial subject. Consider, for example, the

ϕ^4-theory, which is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \ . \tag{4.3}$$

λ represents the strength of the interaction. The equation of motion for the quantum field $\hat{\phi}$ is

$$(\Box + m^2)\hat{\phi}(\mathbf{x}, t) + \frac{\lambda}{3!}\hat{\phi}^3(\mathbf{x}, t) = 0 \ . \tag{4.4}$$

Since this is a nonlinear operator equation, we cannot simply solve it in terms of the annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ as we did in the free field system.

4.2 Green's Function

Let $\hat{H}$ be the total Hamiltonian of the dynamical system of the interacting quantum fields. It consists of the space integral of the Hamiltonian density $\mathcal{H}(\mathbf{x})$. $\hat{H}$ contains not only the free field Hamiltonian $\hat{H}_0$, which we investigated in the previous chapters, but also the interaction term $\hat{V}$ between fields ;

$$\hat{H} = \hat{H}_0 + \hat{V} \ . \tag{4.5}$$

The state which plays a basis in the description of quantum field system is a vacuum $|0\rangle$. It is a normalizable eigenstate of the Hamiltonian $\hat{H}$ with the lowest eigenvalue E_0 ;

$$\hat{H}|0\rangle = E_0|0\rangle \ , \qquad \langle 0|0\rangle = 1 \ . \tag{4.6}$$

We fix the origin of the energy so that $E_0 = 0$. Thus, $\hat{H}|0\rangle = 0$. This means that the vacuum does not receive the time evolution ;

$$e^{-i\hat{H}t}|0\rangle = |0\rangle \ . \tag{4.7}$$

The vacuum is also an eigenstate of the momentum operator $\hat{\mathbf{P}}$ with the vanishing eigenvalue ;

$$\hat{\mathbf{P}}|0\rangle = 0 \ . \tag{4.8}$$

This represents the translational invariance of the vacuum ;

$$e^{-i\mathbf{a}\cdot\hat{\mathbf{P}}}|0\rangle = |0\rangle \ . \tag{4.9}$$

Thus for any 4-dimensional shift vector $l^\mu = (l^0, \mathbf{l})$, $|0\rangle$ satisfies the 4-dimensional translational invariance

$$e^{il\cdot\hat{P}}|0\rangle = |0\rangle \qquad : \ \hat{P}^\mu = (\hat{H}, \ \hat{\mathbf{P}}) \ . \tag{4.10}$$

Green's Function : States other than the vacuum $|0\rangle$ can be constructed by operating various field operators on $|0\rangle$. This means that a matrix element $\langle F|I\rangle$ between any states $|I\rangle$ and $|F\rangle$ is expressed as a matrix element of the product of the field operators between vacuum $\langle 0|$ and $|0\rangle$. The physically promosing products are the time-ordered products. Therefore, all dynamical contents of the quantum system are involved in the vacuum expectation values of the T-product of various field operators :

$$G(x_1, x_2, \cdots, x_N) \equiv \langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_N)]|0\rangle \ , \qquad x_i = x_i^\mu = (x_i^0, \mathbf{x}_i) \ . \tag{4.11}$$

The function $G(x_1, \cdots, x_N)$ is called N-point Green's function.

The translational invariance of the vacuum results in the translational invariance of the Green's functions, as we show below. From (2.37), (2.43) and the commutation $[\hat{H}, \hat{\mathbf{P}}] = 0$, we find

$$e^{il\cdot\hat{P}}\hat{\phi}(x)e^{-il\cdot\hat{P}} = e^{-i\mathbf{l}\cdot\hat{\mathbf{P}}}e^{il^0\hat{H}}\hat{\phi}(\mathbf{x}, t)e^{-il^0\hat{H}}e^{i\mathbf{l}\cdot\hat{\mathbf{P}}} = \hat{\phi}(\mathbf{x}+\mathbf{l}, t+l^0) = \hat{\phi}(x+l) \ . \tag{4.12}$$

Thus, the translational invariance of the vacuum (4.10) gives

$$\begin{aligned} G(x_1, x_2, \cdots, x_N) &= \langle 0|e^{il\cdot\hat{P}}T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)]e^{-il\cdot\hat{P}}|0\rangle \\ &= \langle 0|T[e^{il\cdot\hat{P}}\hat{\phi}(x_1)e^{-il\cdot\hat{P}}\cdots e^{il\cdot\hat{P}}\hat{\phi}(x_N)e^{-il\cdot\hat{P}}]|0\rangle \\ &= \langle 0|T[\hat{\phi}(x_1+l)\cdots\hat{\phi}(x_N+l)]|0\rangle \\ &= G(x_1+l, x_2+l, \cdots, x_N+l) \ . \end{aligned} \tag{4.13}$$

This invariance is reflected in the Fourier transform of $G(x_1, x_2, \cdots, x_N)$ as an appearance of the four-dimensional delta function which represents the conservation of energy and momentum ;

$$\begin{aligned} &\int d^4x_1\cdots d^4x_N\ G(x_1, x_2, \cdots, x_N)e^{i(k_1\cdot x_1+\cdots+k_N\cdot x_N)} \\ &= \int d^4x_1\cdots d^4x_N\ G(x_1-x_N, x_2-x_N, \cdots, 0)e^{i(k_1\cdot x_1+\cdots+k_N\cdot x_N)} \\ &= \int d^4x'_1\cdots d^4x'_{N-1}d^4x_N\ G(x'_1, x'_2, \cdots, 0)e^{i(k_1\cdot x'_1+\cdots+k_{N-1}\cdot x'_{N-1})}e^{i(k_1+\cdots+k_N)x_N} \\ &= (2\pi)^4\delta^4(k_1+\cdots+k_N)\int d^4x'_1\cdots d^4x'_{N-1}\ G(x'_1, x'_2, \cdots, 0)e^{i(k_1\cdot x'_1+\cdots+k_{N-1}\cdot x'_{N-1})} \\ &\equiv (2\pi)^4\delta^4(k_1+\cdots+k_N)\tilde{G}(k_1, \cdots, k_N) \ . \end{aligned} \tag{4.14}$$

where, in the third line, we have changed the integration variables x_i $(i = 1, \cdots, N-1)$ to $x'_i = x_i - x_N$.

The clarification of the further properties of the Green's functions requires the detailed analyses of the dynamics of the quantum field system. Thus the main subject of the quantum field theory is how to elaborate the systematic method which enables us to construct any type of Green's functions.

There are two methods now available. One is the perturbation theory based on the operator formalism in the so-called interaction picture. This method has been well-established through the long history of the quantum field theory. Therefore, it has been widely applied in various fields of the physics. The other is the method based on the path integral representation of the quantum mechanics. Though the history of this method is not so longer than that of the former one, it has been rapidly prevailing over wide regions of physics owing to its remarkable flexibility which comes from the fact that the path integral formulation of the quantum mechanics does not rely on the perturbation expansion of the interaction Hamiltonian $\hat{V}$. Therefore, it often provides us with the technique to explore the nonperturbative behavior the quantum system of the fields reveals.

In this book, we follow the path integral method. In the next chapter, we will give the basic formulation of the path-integral method, and derive the prescription for the perturbation expansion of the Green's function. We will not proceed to the nonperturbative treatment of the quantum field system, which is far beyond the scope of this book.

In order to keep the completeness, we give in the appendix a brief survey of the perturbation theory based on the interaction picture.

Chapter 5

Path Integral

5.1 Path Integral Representation of Quantum Mechanics

Let us consider the quantum system described by the Hamiltanian operator $\hat{H}(\hat{p}, \hat{q})$, where coordinate operator $\hat{q}$ and canonical-conjugate momentum operator $\hat{p}$ are subject to the canonical commutation relation

$$[\hat{q}, \hat{p}] = i \ . \tag{5.1}$$

Let $|q\rangle, |p\rangle$ be the normalized eigenstates of $\hat{q}, \hat{p}$ with eigenvalues q, p, respectively :

$$\hat{q}|q\rangle = q|q\rangle \ , \quad \langle q'|q\rangle = \delta(q' - q) \ , \qquad \int dq \ |q\rangle\langle q| = \hat{1} \tag{5.2}$$

$$\hat{p}|p\rangle = p|p\rangle \ , \quad \langle p'|p\rangle = \delta(p' - p) \ , \qquad \int dp \ |p\rangle\langle p| = \hat{1} \tag{5.3}$$

where $\hat{1}$ is the unit operator. The bracket of $|q\rangle$ and $|p\rangle$ is

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipq} \ , \tag{5.4}$$

which satisfies the property of $\hat{p}$

$$p\langle q|p\rangle = \langle q|\hat{p}|p\rangle = -i\frac{d}{dq}\langle q|p\rangle \tag{5.5}$$

and the normalization condition

$$\langle q'|q\rangle = \int dp' \langle q'|p'\rangle\langle p'|q\rangle = \int \frac{dp'}{2\pi} e^{ip'(q'-q)} = \delta(q' - q) \ . \tag{5.6}$$

Now, we proceed to the Heisenberg picture, and define the Heisenberg operator

$$\hat{q}(t) \equiv e^{i\hat{H}t} \hat{q} e^{-i\hat{H}t} \ . \tag{5.7}$$

We introduce the eigenvector $|q, t\rangle$ of the Heisenberg operator $\hat{q}(t)$ with eigenvalue q by

$$\hat{q}(t)|q, t\rangle = q|q, t\rangle \ . \tag{5.8}$$

Since $qe^{i\hat{H}t}|q\rangle = e^{i\hat{H}t}\hat{q}|q\rangle = e^{i\hat{H}t}\hat{q}e^{-i\hat{H}t}e^{i\hat{H}t}|q\rangle = \hat{q}(t)e^{i\hat{H}t}|q\rangle$, $|q, t\rangle$ is expressed as

$$|q, t\rangle = e^{i\hat{H}t}|q\rangle \ . \tag{5.9}$$

$|q,t\rangle$ produces the completeness equation

$$\int dq\ |q,t\rangle\langle q,t| = e^{i\hat{H}t}\int dq\ |q\rangle\langle q|e^{-i\hat{H}t} = \hat{1}\ . \tag{5.10}$$

Note that $|q,t\rangle$ does not express the time development of $|q\rangle$. The latter is $e^{-i\hat{H}t}|q\rangle$.

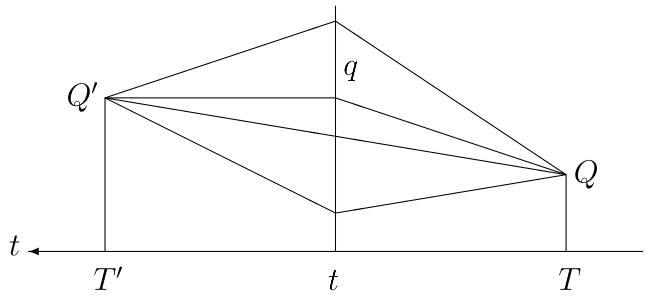
Transition Amplitude : Suppose, at the initial time $t=T$, the state is at an eigenstate of $\hat{q}$ with eigenvalue Q : $\hat{q}|Q\rangle = Q|Q\rangle$. The time development of $|Q\rangle$ is then $e^{-i\hat{H}(t-T)}|Q\rangle$. The transition amplitude for this state to be at an eigenstate $|Q'\rangle$ at the final time $t=T'$ is

$$\langle Q'|e^{-i\hat{H}(T'-T)}|Q\rangle = \langle Q',T'|Q,T\rangle\ . \tag{5.11}$$

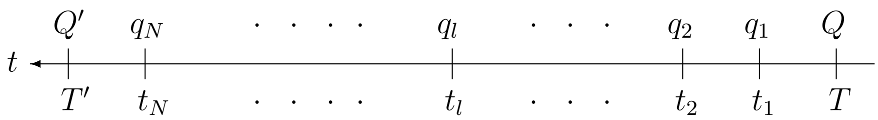
Owing to the completeness equation (5.10), this amplitude is expressed as

$$\langle Q',T'|Q,T\rangle = \int dq\ \langle Q',T'|q,t\rangle\langle q,t|Q,T\rangle\ . \tag{5.12}$$

Here, we put a constraint on t so that $T' > t > T$. The equation (5.12) shows that the transition amplitude from T to T' is decomposed to the product of two transition amplitudes, one from T to the transit time t, and the other, from t to T', integrated over all possible transit points q.



Let us insert the completeness equation N times at times t_l $(l=1,\cdots,N)$ with an equal time spacing $t_l - t_{l-1} = \Delta t$.



The transition amplitude is expressed as

$$\begin{aligned}\langle Q',T'|Q,T\rangle &= \int\prod_{l=1}^{N} dq_l\langle Q',T'|q_N,t_N\rangle\langle q_N,t_N|\cdots \\ &\quad\times\cdots|q_l,t_l\rangle\langle q_l,t_l|q_{l-1},t_{l-1}\rangle\langle q_{l-1},t_{l-1}|\cdots|q_1,t_1\rangle\langle q_1,t_1|Q,T\rangle\ .\end{aligned} \tag{5.13}$$

Each fraction of this amplitude is written, inserting the completeness $\int dp|p\rangle\langle p| = \hat{1}$, as

$$\langle q_l,t_l|q_{l-1},t_{l-1}\rangle = \langle q_l|e^{-i\hat{H}\Delta t}|q_{l-1}\rangle = \int dp_l\langle q_l|p_l\rangle\langle p_l|e^{-i\hat{H}\Delta t}|q_{l-1}\rangle \tag{5.14}$$

Now we consider the limit $N\to\infty$ and $\Delta t\to 0$. In this limit, the second factor of the integrand is approximated as

$$\begin{aligned}\langle p_l|e^{-i\hat{H}(\hat{p},\hat{q})\Delta t}|q_{l-1}\rangle &\simeq \langle p_l|1 - i\hat{H}(\hat{p},\hat{q})\Delta t|q_{l-1}\rangle \\ &= \langle p_l|1 - iH(p_l,q_{l-1})\Delta t|q_{l-1}\rangle \simeq e^{-iH(p_l,q_{l-1})\Delta t}\langle p_l|q_{l-1}\rangle\ ,\end{aligned} \tag{5.15}$$

where we assumed that, when $\hat{H}(\hat{p},\hat{q})$ contains the product of $\hat{q}$ and $\hat{p}$, $\hat{H}$ has been arranged, in advance, so that $\hat{p}$ stands the left of $\hat{q}$ using the commutator. Due to the formula (5.4), (5.14) turns out to

$$\langle q_l, t_l | q_{l-1}, t_{l-1}\rangle = \int \frac{dp_l}{2\pi} \ \exp\left[i\Delta t \left(p_l \frac{q_l - q_{l-1}}{\Delta t} - H(p_l, q_{l-1}) \right) \right] \ . \tag{5.16}$$

Thus, the full transition amplitude (5.13) is expressed as

$$\langle Q', T' | Q, T\rangle = \lim_{N\to\infty} \int [dqdp] \ \exp\left[i\Delta t \sum_{l=1}^{N+1} \left(p_l \frac{q_l - q_{l-1}}{\Delta t} - H(p_l, q_{l-1}) \right) \right] \ , \tag{5.17}$$

where $q_0 \equiv Q$, $q_{N+1} \equiv Q'$ and

$$[dqdp] \equiv \prod_{l=1}^{N} dq_l \prod_{l=1}^{N+1} \frac{dp_l}{2\pi} \ . \tag{5.18}$$

In the continuous expression, it takes a form

$$\langle Q', T' | Q, T\rangle = \int [dqdp] \ \exp \ i \int_T^{T'} dt \ [\ p(t)\dot{q}(t) - H(p(t), q(t)) \ ] \ . \tag{5.19}$$

This expression claims that the transition amplitude is obtained by integrating a phase factor

$$\exp i \int_T^{T'} dt \ [\ p(t)\dot{q}(t) - H(p(t), q(t)) \ ] \tag{5.20}$$

over all possible path $q(t)$ and $p(t)$ under the constraints

$$q(t = T) = Q \ , \qquad q(t = T') = Q' \ . \tag{5.21}$$

Thus, this is called “path integral”.

Matrix Element of Heisenberg Operator : Suppose $\hat{q}(t_a)$ is the Heisenberg operator of time t_a with $T' > t_a > T$. The matrix element of $\hat{q}(t_a)$ between $|Q, T\rangle$ and $|Q', T'\rangle$ is calculated to be

$$\begin{aligned} \langle Q', T' | \hat{q}(t_a) | Q, T\rangle &= \langle Q', T' | \int dq_a | q_a, t_a\rangle\langle q_a, t_a | \hat{q}(t_a) | Q, T\rangle \\ &= \int dq_a \ q_a \langle Q', T' | q_a, t_a\rangle\langle q_a, t_a | Q, T\rangle \\ &= \int [dqdp] q(t_a) \exp \ i \int_T^{T'} dt \ [\ p(t)\dot{q}(t) - H(p(t), q(t)) \ ] \ , \end{aligned} \tag{5.22}$$

where $q(t_a) \equiv q_a$. This is only the insertion of $q(t_a)$ into the integrand of (5.19). Next, we calculate the matrix element of $\hat{q}(t_a)\hat{q}(t_b)$. We assume $T' > t_a > t_b > T$. The result is

$$\begin{aligned} \langle Q', T' | \hat{q}(t_a)\hat{q}(t_b) | Q, T\rangle &= \int dq_a dq_b \ \langle Q', T' | q_a, t_a\rangle\langle q_a, t_a | \hat{q}(t_a) | q_b, t_b\rangle\langle q_b, t_b | \hat{q}(t_b) | Q, T\rangle \\ &= \int dq_a dq_b \ q_a q_b \langle Q', T' | q_a, t_a\rangle\langle q_a, t_a | q_b, t_b\rangle\langle q_b, t_b | Q, T\rangle \\ &= \int [dqdp] \ q(t_a) q(t_b) \exp \ i \int_T^{T'} dt \ [\ p(t)\dot{q}(t) - H(t) \ ] \ , \end{aligned} \tag{5.23}$$

where $H(t) \equiv H(p(t), q(t))$. Since, $q(t_a)$ and $q(t_b)$ in the integrand are commutative, this integral is well-defined independently of the ordering of t_a and t_b. If we take the matrix element of $\hat{q}(t_b)\hat{q}(t_a)$ with $T' > t_b > t_a > T$, the result is again the right-hand-side of (5.23). This means that the integral in the right-hand-side in fact represents the matrix element of $T[\hat{q}(t_a)\hat{q}(t_b)]$:

$$\langle Q', T'|T[\hat{q}(t_a)\hat{q}(t_b)]|Q, T\rangle = \int [dqdp]\ q(t_a)q(t_b) \exp\ i\int_T^{T'} dt\ [\ p(t)\dot{q}(t) - H(t)\ ]\ . \tag{5.24}$$

Generalizing this consideration, we realize, for $T' > t_i\ (i = 1, \cdots, n) > T$, that

$$\langle Q', T'|T[\hat{q}(t_1)\cdots\hat{q}(t_n)]|Q, T\rangle = \int [dqdp]\ q(t_1)\cdots q(t_n) \exp\ i\int_T^{T'} dt\ [\ p(t)\dot{q}(t) - H(t)\ ]\ . \tag{5.25}$$

Generating Functional : Let $J(t)$ be an arbitrary function of t, and take the following matrix element based on the formula (5.25) ;

$$\begin{aligned}
Z[J] &\equiv \langle Q', T'|T\left[\exp\ i\int_T^{T'} dtJ(t)\hat{q}(t)\right]|Q, T\rangle \\
&= \langle Q', T'|T\left[\sum_{M=0}^{\infty}\frac{i^M}{M!}\int_T^{T'} dt_1 J(t_1)\hat{q}(t_1)\cdots\int_T^{T'} dt_M J(t_M)\hat{q}(t_M)\right]|Q, T\rangle \\
&= \int [dqdp]\sum_{M=0}^{\infty}\frac{i^M}{M!}\int_T^{T'} dt_1 J(t_1)q(t_1)\cdots\int_T^{T'} dt_M J(t_M)q(t_M)\ \exp\ i\int_T^{T'} dt\ [\ p(t)\dot{q}(t) - H(t)\ ] \\
&= \int [dqdp] \exp\ i\int_T^{T'} dt\ [\ p(t)\dot{q}(t) - H(t) + J(t)q(t)\ ]\ .
\end{aligned} \tag{5.26}$$

$Z[J]$ is called generating functional. It is a functional of the so-called source function $J(t)$ of $\hat{q}(t)$. Now, we define the functional derivative of any functional $F[J]$ of $J(t)$ by

$$\frac{\delta F[J]}{\delta J(t_1)} \equiv \lim_{\epsilon\to 0}\frac{F[J(t) + \epsilon\delta(t - t_1)] - F[J(t)]}{\epsilon}\ . \tag{5.27}$$

From this definition, we find

$$\frac{\delta J(t_2)}{\delta J(t_1)} = \delta(t_2 - t_1)\ . \tag{5.28}$$

Let us take the functional derivative of $Z[J]$ with respect to $J(t_1)$ with $T' > t_1 > T$. From the first expression of (5.26), we find

$$\begin{aligned}
\frac{\delta Z[J]}{\delta J(t_1)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle Q', T'|T\left[\exp\ i\int_T^{T'} dt(J(t) + \epsilon\delta(t - t_1))\hat{q}(t) - \exp\ i\int_T^{T'} dtJ(t)\hat{q}(t)\right]|Q, T\rangle \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle Q', T'|T\left[(1 + i\epsilon\hat{q}(t_1))\exp\ i\int_T^{T'} dtJ(t)\hat{q}(t) - \exp\ i\int_T^{T'} dtJ(t)\hat{q}(t)\right]|Q, T\rangle \\
&= \langle Q', T'|T\left[i\hat{q}(t_1)\exp\ i\int_T^{T'} dtJ(t)\hat{q}(t)\right]|Q, T\rangle\ .
\end{aligned} \tag{5.29}$$

It is obvious that the N-th derivative of $Z[J]$ with respect to $J(t_i)$ $(T' > t_1, \cdots, t_N > T)$ followed by setting $J(t) = 0$ gives

$$\left.\frac{\delta^N Z[J]}{\delta J(t_1)\cdots\delta J(t_N)}\right|_{J=0} = i^N\langle Q', T'|T[\hat{q}(t_1)\cdots\hat{q}(t_N)]|Q, T\rangle\ . \tag{5.30}$$

Extension to Many-Degrees of Freedom : Up to present, we have considered the quantum system with single dynamical freedom described by the set $\hat{q}$ and $\hat{p}$. The extension to many dynamical freedom $\hat{q}_\alpha$, $\hat{p}_\alpha$ $(\alpha = 1, \cdots, n)$ is straightforward :

$$\hat{\mathbf{q}} = \{\hat{q}_\alpha\} \ , \quad \hat{\mathbf{p}} = \{\hat{p}_\alpha\} \ , \qquad [\hat{q}_\alpha, \hat{p}_\beta] = i\delta_{\alpha\beta} \quad \alpha, \beta = 1, \cdots, n \ , \tag{5.31}$$

$$\hat{\mathbf{q}}|\mathbf{q}\rangle = \mathbf{q}|\mathbf{q}\rangle \ , \qquad \langle \mathbf{q}'|\mathbf{q}\rangle = \delta^n(\mathbf{q}' - \mathbf{q}) \ , \quad \int d^n\mathbf{q} \ |\mathbf{q}\rangle\langle\mathbf{q}| = \hat{1} \ , \tag{5.32}$$

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \ , \qquad \langle \mathbf{p}'|\mathbf{p}\rangle = \delta^n(\mathbf{p}' - \mathbf{p}) \ , \quad \int d^n\mathbf{p} \ |\mathbf{p}\rangle\langle\mathbf{p}| = \hat{1} \ , \tag{5.33}$$

$$\langle \mathbf{q}|\mathbf{p}\rangle = \frac{1}{(2\pi)^{n/2}} e^{i\mathbf{p}\cdot\mathbf{q}} \ , \qquad \mathbf{p}\cdot\mathbf{q} \equiv \sum_{\alpha=1}^{n} p_\alpha q_\alpha \ . \tag{5.34}$$

The path integral representation of the transition amplitude is

$$\begin{aligned} \langle \mathbf{Q}', T'|\mathbf{Q}, T\rangle &= \lim_{N\to\infty} \int [d^n\mathbf{q} d^n\mathbf{p}] \exp\left[i\Delta t \sum_{l=1}^{N+1} \left[\frac{\mathbf{p}_l \cdot (\mathbf{q}_l - \mathbf{q}_{l-1})}{\Delta t} - H(\mathbf{p}_l, \mathbf{q}_{l-1}) \right]\right] \\ &\equiv \int [dqdp] \exp \ i\int_T^{T'} dt \ [\ \mathbf{p}(t)\cdot\dot{\mathbf{q}}(t) - H(\mathbf{p}(t), \mathbf{q}(t)) \ ] \ , \end{aligned} \tag{5.35}$$

where

$$[d^n\mathbf{q} d^n\mathbf{p}] \equiv \prod_{l=1}^{N} d^n\mathbf{q}_l \prod_{l=1}^{N+1} \frac{d^n\mathbf{p}_l}{2\pi} \ . \tag{5.36}$$

5.2 Path Integral Representation of Quantum Field Theory

Let us start with the scalar field. Following the lattice discretization method given in II.2, we represent the field operator $\hat{\phi}(\mathbf{x})$ and its canonical-conjugate momentum density $\hat{\pi}(\mathbf{x})$ by the discretized operators ;

$$\hat{\phi}_{\boldsymbol{\alpha}} \ , \quad \hat{p}_{\boldsymbol{\alpha}} \equiv \epsilon^3 \hat{\pi}_{\boldsymbol{\alpha}} \ , \qquad [\hat{\phi}_{\boldsymbol{\alpha}}, \hat{p}_{\boldsymbol{\beta}}] = i\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} \ . \tag{5.37}$$

The Hamiltonian is expressed as

$$\hat{H} = \int d^3\mathbf{x} \ \hat{\mathcal{H}} \longleftrightarrow \sum_{\boldsymbol{\alpha}} \epsilon^3 \hat{\mathcal{H}}_{\boldsymbol{\alpha}}(\hat{\pi}, \hat{\phi}) \ . \tag{5.38}$$

Now, we introduce the eigenstates of $\hat{\phi}$ and $\hat{\pi}$;

$$\hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle \longleftrightarrow \hat{\phi}_{\boldsymbol{\alpha}}|\phi\rangle = \phi_{\boldsymbol{\alpha}}|\phi\rangle \tag{5.39}$$

$$\hat{\pi}(\mathbf{x})|\pi\rangle = \pi(\mathbf{x})|\pi\rangle \longleftrightarrow \hat{\pi}_{\boldsymbol{\alpha}}|\pi\rangle = \pi_{\boldsymbol{\alpha}}|\pi\rangle \ . \tag{5.40}$$

Applying the expression (5.35), the transition amplitude from the eigenstate $\hat{\phi}(\mathbf{x})|\Phi\rangle = \Phi(\mathbf{x})|\Phi\rangle$ at $t = T$ to the eigenstate $\hat{\phi}(\mathbf{x})|\Phi'\rangle = \Phi'(\mathbf{x})|\Phi'\rangle$ at $t = T'$ is expressed as

$$\begin{aligned} \langle \Phi', T'|\Phi, T\rangle &= \lim_{N\to\infty} \int \prod_{l=1}^{N} \left(\prod_{\boldsymbol{\alpha}} d\phi_{\boldsymbol{\alpha}}(t_l) \right) \prod_{l=1}^{N+1} \left(\prod_{\boldsymbol{\alpha}} \frac{\epsilon^3 d\pi_{\boldsymbol{\alpha}}(t_l)}{2\pi} \right) \\ &\qquad \times \exp\left[i\Delta t \sum_{l=1}^{N} \sum_{\boldsymbol{\alpha}} \epsilon^3 \left[\pi_{\boldsymbol{\alpha}}(t_l) \frac{\phi_{\boldsymbol{\alpha}}(t_l) - \phi_{\boldsymbol{\alpha}}(t_{l-1})}{\Delta t} - \mathcal{H}_{\boldsymbol{\alpha}}(\pi, \phi) \right]\right] \\ &\equiv \int [d\phi d\pi] \ \exp \ i\int_T^{T'} d^4x \left[\ \pi(\mathbf{x}, t)\dot{\phi}(\mathbf{x}, t) - \mathcal{H}(\pi, \phi) \ \right] \ . \end{aligned} \tag{5.41}$$

The integral $[d\phi d\pi]$ now means to integrate over all possible 4-dimensional functional form of $\phi(x) \equiv \phi(\mathbf{x}, t)$ and $\pi(x) \equiv \pi(\mathbf{x}, t)$ between the time interval $T < t < T'$, under the constraint of $\phi(\mathbf{x}, T) = \Phi(\mathbf{x})$ and $\phi(\mathbf{x}, T') = \Phi'(\mathbf{x})$.

Generating Functional of Green Function : According to the expression (5.26), we define the functional of $J(x) \equiv J(\mathbf{x}, t)$ by

$$\begin{aligned} Z[J] &\equiv \langle \Phi', T' | T[\exp\ i\int_T^{T'} d^4x\ J(x)\hat{\phi}(x)] | \Phi, T\rangle \\ &= \int [d\phi d\pi]\ \exp\ i\int_T^{T'} d^4x\ \Big[\ \pi(x)\dot{\phi}(x) - \mathcal{H}(x) + J(x)\phi(x)\ \Big]\ . \end{aligned} \tag{5.42}$$

Since, $Z[J]$ is a functional of 4-dimensional function $J(x)$, the functional derivative is now defined by

$$\frac{\delta J(x_1)}{\delta J(x_2)} = \delta^4(x_1 - x_2) \equiv \delta(t_1 - t_2)\delta^3(\mathbf{x}_1 - \mathbf{x}_2)\ . \tag{5.43}$$

Then, by the similar procedure to that of (5.30), we find

$$\frac{1}{Z[J]}\frac{\delta^N Z[J]}{\delta J(x_1)\cdots\delta J(x_N)}\bigg|_{J=0} = i^N \frac{\langle \Phi', T' | T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)] | \Phi, T\rangle}{\langle \Phi', T' | \Phi, T\rangle}\ , \tag{5.44}$$

where each time $x_i^0 \equiv t_i$ $(i = 1, \cdots, N)$ is restricted to $T < t_i < T'$.

Limit $\mathbf{T \to -\infty,\ T' \to \infty}$: We wish to remove the artificial constraint $T < t_i < T'$ by taking the limit $T \to -\infty,\ T' \to \infty$ so that all of t_i can take an arbitrary value. In order to examine the limiting behavior, let us introduce the eigenvectors $|n\rangle$ of the Hamiltonian $\hat{H}$ by

$$\hat{H}|n\rangle = E_n|n\rangle \quad n = 0, 1, 2, \cdots, \infty\ , \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{1}\ . \tag{5.45}$$

Note that the vacuum $|0\rangle$ has vanishing energy $E_0 = 0$. The denominator of (5.44) is expressed as

$$\begin{aligned} \langle \Phi', T' | \Phi, T\rangle &= \langle \Phi' | e^{-i\hat{H}(T'-T)} | \Phi\rangle = \langle \Phi' | e^{-i\hat{H}(T'-T)} \sum_{n=0}^{\infty} |n\rangle\langle n|\Phi\rangle \\ &= \sum_{n=0}^{\infty} e^{-iE_n(T'-T)} \langle \Phi' | n\rangle\langle n | \Phi\rangle\ . \end{aligned} \tag{5.46}$$

Similarly, the numerator is

$$\langle \Phi', T' | T[\hat{\phi}\cdots\hat{\phi}] | \Phi, T\rangle = \sum_{m,n=0}^{\infty} e^{-i(E_m T' - E_n T)} \langle \Phi' | m\rangle\langle m | T[\hat{\phi}\cdots\hat{\phi}] | n\rangle\langle n | \Phi\rangle\ . \tag{5.47}$$

We find that the both functions show despairingly rapid oscillation in the limit $T \to -\infty,\ T' \to \infty$, and we cannot obtain the well-defined limit. However, we notice here a "miraculous"fact. Replace T and T' in these expressions by $T(1-i\epsilon)$ and $T'(1-i\epsilon)$ with an infinitesimal positive ϵ. Owing to the presence of $-i\epsilon$, the intermediate states $|n\rangle$ and $|m\rangle$ other than the vacuum $|0\rangle$ drop out in the limit $T \to -\infty,\ T' \to \infty$, because $E_m > 0$ $(m > 0)$. Thus, the both functions have well-defined limits and their ratio (5.44) in this limit turns out to

$$\begin{aligned} \frac{\langle \Phi', T'(1-i\epsilon) | T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)] | \Phi, T(1-i\epsilon)\rangle}{\langle \Phi', T'(1-i\epsilon) | \Phi, T(1-i\epsilon)\rangle} &\underset{\substack{T\to-\infty \\ T'\to+\infty}}{\longrightarrow} \frac{\langle \Phi' | 0\rangle\langle 0 | T[\hat{\phi}\cdots\hat{\phi}] | 0\rangle\langle 0 | \Phi\rangle}{\langle \Phi' | 0\rangle\langle 0 | \Phi\rangle} \\ &= \langle 0 | T[\hat{\phi}\cdots\hat{\phi}] | 0\rangle\ . \end{aligned} \tag{5.48}$$

This is precisely the Green's function (4.11). Notice that this limit is independent not only of T and T', but also of Φ and Φ'. In this way, we have got a valuable formula for the Green's function $G(x_1, \cdots, x_N)$:

$$G(x_1, \cdots, x_N) \equiv \langle 0|T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)]|0\rangle = \frac{1}{Z[J]}\frac{1}{i^N}\frac{\delta^N Z[J]}{\delta J(x_1)\cdots\delta J(x_N)}\bigg|_{J=0} , \tag{5.49}$$

$$\begin{aligned} Z[J] &= \int [d\phi d\pi] \exp\, i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} d^4x \, \left[\, \pi\dot{\phi} - \mathcal{H} + J\phi \,\right] \\ &= Z[0]\sum_{N=0}^{\infty}\frac{i^N}{N!}\int d^4x_1\cdots d^4x_N J(x_1)\cdots J(x_N) G(x_1,\cdots,x_N) \, . \end{aligned} \tag{5.50}$$

[$d\pi$] Integration : Usually, Hamiltonian density $\mathcal{H}$ contains $\pi(x)$ at most in the quadratic form. Therefore, the path integral over $\pi(x)$ becomes a Gaussian integral, and explicit integration is possible.

For example, the Hamiltonian of the scaler field is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \, , \tag{5.51}$$

and the integrand in the exponent of (5.50) becomes

$$\begin{aligned} \pi\dot{\phi} - \mathcal{H} &= -\frac{1}{2}(\pi-\dot{\phi})^2 + \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \\ &= -\frac{1}{2}(\pi-\dot{\phi})^2 + \mathcal{L} \, , \end{aligned} \tag{5.52}$$

$$\mathcal{L} \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \, . \tag{5.53}$$

$\mathcal{L}$ is just the Lagrangian of the scalar field theory. Now, we shift the integration variable π to $\pi' = \pi - \dot{\phi}$. Since the integration measure is invaliant under this shift, $[d\pi'] = [d\pi]$, the pass integral becomes

$$\begin{aligned} Z[J] &= \int [d\phi d\pi'] \, \exp\, i\int d^4x \, \left[\, -\frac{1}{2}(\pi')^2 + \mathcal{L} + J\phi \,\right] \\ &\propto \int [d\phi] \, \exp\, i\int d^4x \, [\, \mathcal{L} + J\phi \,] \, . \end{aligned} \tag{5.54}$$

Thus, $Z[J]$ is expressed, upto the J independent factor, in the form

$$Z[J] = \int [d\phi] \, \exp\, i\int_{-\infty(1-i\epsilon)}^{+\infty(1-i\epsilon)} d^4x \, [\, \mathcal{L} + J\phi \,] \, . \tag{5.55}$$

Classical Limit : Let us return, for a while, to the expression of the transition amplitude (5.41) for a scaler field ϕ and calculate a matrix element of $\hat{\phi}(x)$, recovering the Planck constant $\hbar$. After the completition of the $[d\pi]$ integration, we obtain

$$\langle \Phi', T'|\hat{\phi}(x)|\Phi, T\rangle \propto \int [d\phi]\phi(x) \exp\frac{i}{\hbar}\int_T^{T'} d^4x\, \mathcal{L} \equiv \int [d\phi]\phi(x)\exp\frac{i}{\hbar}I[\phi] \, . \tag{5.56}$$

The classical limit of the quantum dynamics is recovered by $\hbar \to 0$. At this limit, the field configuration which gives the dominant contribution to the $[d\phi]$ integration is the classical configuration

$\phi_{\rm cl}(x)$ which keeps the classical action integral $I[\phi]$ stationary, $\delta I[\phi_{\rm cl}] = 0$. The configurations other than $\phi_{\rm cl}(x)$, integrated over $[d\phi]$, cancel out owing to the rapid phase rotation of $e^{\frac{i}{\hbar}I[\phi]}$. Thus the matrix element of $\hat{\phi}(x)$ is frozen to $\phi_{\rm cl}(x)$. This is just a classical action principle.

iϵ Prescription : The result (5.48) shows that the generating functional $Z[J]$ which has well-defined value in the limit $|T|, |T'| \to \infty$ generates the desirable Green's functions. The prescription adopted there is equivalent to the replacement of the Hamiltonian $\hat{H} \to (1 - i\epsilon)\hat{H}$. This suggests an alternative prescription. We may add $-\frac{1}{2}i\epsilon\hat{\phi}^2$ to the Hamiltonian density $\hat{\mathcal{H}}$. This will equally work as long as $\langle 0|\hat{\phi}^2|0\rangle < \langle n|\hat{\phi}^2|n\rangle$ $(n \neq 0)$ is satisfied. This leads to the replacement of the Lagrangian $\mathcal{L}$ in (5.55) by $\mathcal{L} + \frac{1}{2}i\epsilon\phi^2$, and we have

$$Z[J] = \int [d\phi] \exp i \int_{-\infty}^{+\infty} d^4x \left[\mathcal{L} + \frac{1}{2}i\epsilon\phi^2 + J\phi\right] . \tag{5.57}$$

Notice that, owing to the $i\epsilon$ term, the $[d\phi]$ integration converges at the limit $|\phi| \to \infty$. We will shortly find that this prescription correctly reproduces the scalar propagator $\Delta_F(x - y)$.

The convergence of the path integral validates the partial integration

$$\int [d\phi] \frac{\delta}{\delta\phi(x)} \exp i \int d^4x \left[\mathcal{L} + \frac{1}{2}i\epsilon\phi^2 + J\phi\right] = 0 . \tag{5.58}$$

$\mathbf{Z_0}$[J] of Real Free Scalar Field : Let us determine the functional form of the generating functional of the real free scalar field $Z_0[J]$;

$$\begin{aligned} Z_0[J] &\equiv \int [d\phi] \exp i \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}i\epsilon\phi^2 + J\phi\right] \\ &= \int [d\phi] \exp i \int d^4x \left[\frac{1}{2}\phi(-\Box - m^2 + i\epsilon)\phi + J\phi\right] . \end{aligned} \tag{5.59}$$

For the $[d\phi]$ integration, it is convenient to use a vector and matrix notation

$$\phi \equiv \phi_x \ : \ \text{column vector} , \qquad \phi^T \ : \ \text{row vector} \tag{5.60}$$

$$(-\Box - m^2 + i\epsilon)_{xy} \equiv (-\Box - m^2 + i\epsilon)\delta^4(x - y) \ : \ \text{matrix} . \tag{5.61}$$

Then, the 4-dimensional integral in the exponent of (5.59) is expressed as

$$\begin{aligned} &\frac{1}{2}\phi^T(-\Box - m^2 + i\epsilon)\phi + J^T\phi \\ &= \frac{1}{2}\left(\phi + (-\Box - m^2 + i\epsilon)^{-1}J\right)^T(-\Box - m^2 + i\epsilon)\left(\phi + (-\Box - m^2 + i\epsilon)^{-1}J\right) \\ &\quad -\frac{1}{2}J^T(-\Box - m^2 + i\epsilon)^{-1}J . \end{aligned} \tag{5.62}$$

Now, we shift the integration variable ϕ to

$$\phi' = \phi + (-\Box - m^2 + i\epsilon)^{-1}J , \tag{5.63}$$

which preserves the integration measure invariant, $[d\phi'] = [d\phi]$. The $[d\phi']$ integration gives $Z_0[J]$ only the J independent factor $\int [d\phi'] \exp i\frac{1}{2}\phi'^T(-\Box - m - 2 + i\epsilon)\phi'$. We disregard this factor.

The matrix $(-\Box - m^2 + i\epsilon)^{-1}$ in the second term of (5.62), in the continuous expression, takes the form

$$\begin{aligned}(-\Box - m^2 + i\epsilon)^{-1}_{xy} &\equiv (-\Box - m^2 + i\epsilon)^{-1}\delta^4(x-y) \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-y)} \\ &= \Delta_F(x-y)\ , \end{aligned} \tag{5.64}$$

which is just the Feynman propagator. Thus, we obtain the expression of $Z_0[J]$;

$$Z_0[J] = e^{-\frac{1}{2}\int d^4x d^4y J(x) i\Delta_F(x-y)J(y)}\ . \tag{5.65}$$

Problem (5.2-1) Based on the partial integrability

$$\int [d\phi]\frac{\delta}{\delta\phi(x)}\phi(y)\exp i\int d^4x\left[\frac{1}{2}\phi(-\Box - m^2 + i\epsilon)\phi - V(\phi)\right] = 0\ , \tag{5.66}$$

derive

$$(\Box + m^2)\langle 0|T[\hat\phi(x)\hat\phi(y)]|0\rangle + \langle 0|T\left[V'(\hat\phi(x))\ \hat\phi(y)\right]|0\rangle = -i\delta^4(x-y)\ , \tag{5.67}$$

where $V' = dV/d\phi$. Confirm the reason why $\Box$ operates on the matrix element instead of directly operating to $\hat\phi(x)$.

5.3 Perturbation Theory

Now we proceed to the quantum system of interacting field. Let the Lagrangian be given by

$$\mathcal{L} = \mathcal{L}_0(\phi) + \mathcal{L}_{\rm int}(\phi)\ . \tag{5.68}$$

$\mathcal{L}_0$ is a free Lagrangian and $\mathcal{L}_{\rm int}$ contains interaction terms of ϕ. The generating functional is

$$Z[J] = \int [d\phi]\exp\ i\int d^4x\,[\ \mathcal{L}(x) + J(x)\phi(x)\ ]\ , \tag{5.69}$$

where the $i\epsilon$ term is tacitly assumed. According to the definition of the functional derivative (5.43), we have

$$\frac{1}{i}\frac{\delta}{\delta J(y)}\exp i\int d^4x\ J(x)\phi(x) = \phi(y)\exp i\int d^4x\ J(x)\phi(x)\ . \tag{5.70}$$

Therefore, (5.69) can be rewritten in the form

$$\begin{aligned} Z[J] &= \int [d\phi]\exp i\int d^4y\ \mathcal{L}_{\rm int}\left(\frac{\delta}{i\delta J(y)}\right)\exp i\int d^4x\ [\ \mathcal{L}_0 + J\phi\ ] \\ &= \exp i\int d^4y\ \mathcal{L}_{\rm int}\left(\frac{\delta}{i\delta J(y)}\right)Z_0[J]\ , \end{aligned} \tag{5.71}$$

with $Z_0[J]$ given by (5.65). For the actual calculation of (5.71), it is convenient to introduce a following trick. $Z_0[J]$ can be expressed as

$$Z_0[J] = Z_0\left[\frac{\delta}{i\delta\phi}\right]\exp i\int d^4x\ J\phi\Big|_{\phi=0}\ . \tag{5.72}$$

Substituting this expression for $Z_0[J]$ in (5.71) and shifting $Z_0[\delta/i\delta\phi]$ to the head, we have

$$\begin{aligned} Z[J] &= Z_0\left[\frac{\delta}{i\delta\phi}\right] \exp i\int d^4x\ [\ \mathcal{L}_{\rm int}(\phi) + J\phi\ ]\Big|_{\phi=0} \\ &= \exp\left[\frac{1}{2}\int d^4x d^4y \frac{\delta}{\delta\phi(x)} i\varDelta_F(x-y)\frac{\delta}{\delta\phi(y)}\right]\ \exp i\int d^4x\ [\mathcal{L}_{\rm int}(\phi) + J\phi]\Big|_{\phi=0}\ . \end{aligned} \tag{5.73}$$

This expression enables us to calculate $Z[J]$ in the perturbation series of $\mathcal{L}_{\rm int}(\phi)$;

$$Z[J] = \exp\left[\frac{1}{2}\int d^4x d^4y \frac{\delta}{\delta\phi(x)} i\varDelta_F(x-y)\frac{\delta}{\delta\phi(y)}\right]\ \sum_{M=0}^{\infty}\frac{1}{M!}\left[i\int d^4x\ \mathcal{L}_{\rm int}(\phi)\right]^M \exp i\int d^4x\ J\phi\Big|_{\phi=0}\ . \tag{5.74}$$

Wick's Expansion Theorem : Let us introduce the notation

$$\begin{aligned} \langle F[\phi]\rangle_0 &= \exp\left[\frac{1}{2}\int d^4x d^4y \frac{\delta}{\delta\phi(x)} i\varDelta_F(x-y)\frac{\delta}{\delta\phi(y)}\right] F[\phi]\Big|_{\phi=0} \\ &\equiv e^{\frac{1}{2}\delta i\varDelta\delta}F[\phi]\Big|_0\ . \end{aligned} \tag{5.75}$$

In the simple case of $F[\phi] = \phi(x_1)\phi(x_2)$, only the first term $\frac{1}{2}\delta i\varDelta\delta$ in the Taylor expansion of $\exp[\frac{1}{2}\delta i\varDelta\delta]$ survives after setting $\phi = 0$. Thus,

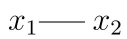
$$\begin{aligned} \langle\phi(x_1)\phi(x_2)\rangle_0 &= e^{\frac{1}{2}\delta i\varDelta\delta}\phi(x_1)\phi(x_2)\Big|_0 = \frac{1}{2}\delta i\varDelta\delta\ \phi(x_1)\phi(x_2) \\ &= \frac{1}{2}\int d^4x d^4y \frac{\delta}{\delta\phi(x)} i\varDelta_F(x-y)\frac{\delta}{\delta\phi(y)}\ \phi(x_1)\phi(x_2) \\ &= \frac{1}{2}\int d^4x d^4y\ i\varDelta_F(x-y)\left(\delta^4(x_1-x)\delta^4(x_2-y) + \delta^4(x_1-y)\delta^4(x_2-x)\right) \\ &= \frac{1}{2}\left(i\varDelta_F(x_1-x_2) + i\varDelta_F(x_2-x_1)\right) \\ &= i\varDelta_F(x_1-x_2)\ . \end{aligned} \tag{5.76}$$

In the case of $F[\phi] = \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)$, we need the second expansion term,

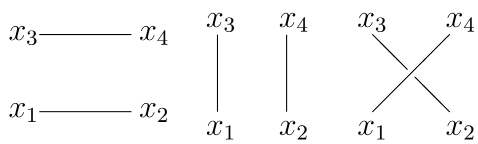
$$\begin{aligned} \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_0 &= e^{\frac{1}{2}\delta i\varDelta\delta}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big|_0 \\ &= \frac{1}{2!}\cdot\frac{1}{2}\delta i\varDelta\delta\cdot\frac{1}{2}\delta i\varDelta\delta\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\ &= \ \ i\varDelta_F(x_1-x_2)i\varDelta_F(x_3-x_4) \\ &\quad + i\varDelta_F(x_1-x_3)i\varDelta_F(x_2-x_4) \\ &\quad + i\varDelta_F(x_1-x_4)i\varDelta_F(x_2-x_3)\ . \end{aligned} \tag{5.77}$$

Let us abbreviate these results in the form

$$\langle\phi_1\phi_2\rangle_0 = \overline{\phi_1\phi_2}\ , \tag{5.78}$$



$$\langle\phi_1\phi_2\phi_3\phi_4\rangle_0 = \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4\ . \tag{5.79}$$



The symbol

$$\overbracket{\phi_i\ \phi_j} = i\Delta_F(x_i - x_j) \tag{5.80}$$

is called Wick's "contraction".

Since, the Tayler expansions of $\exp[\frac{1}{2}\delta i\Delta\delta]$ always contain even number of differentials $\delta/\delta\phi$, $\langle F[\phi]\rangle_0$ for odd polynomial of ϕ simply vanishes ;

$$\langle\phi_1 \cdots \phi_{2n+1}\rangle_0 = 0\ . \tag{5.81}$$

For even polynomials, we have

$$\langle\phi_1\phi_2\cdots\phi_{2n}\rangle_0 = \frac{1}{n!}\left[\frac{1}{2}\delta i\Delta\delta\right]^n \phi_1\phi_2\cdots\phi_{2n}\ . \tag{5.82}$$

The operation of $2n$ differentials gives various patterns of contractions. Let us look at some specific pattern of contractions. It contains n contraction lines. Since we have n double-differential operators $\frac{1}{2}\delta i\Delta\delta$, we have $n!$ possibilities to assign n double-differential operators to n contraction lines. Thus, the factor $1/n!$ in (5.82) is cancelled. The factor 1/2 in the double-differential operator is always cancelled as we found in (5.76). Thus, we arrive at a very simple result, which is called Wick's expansion theorem ;

$$\langle\phi_1\phi_2\cdots\phi_{2n}\rangle_0 = \sum \overbracket{\phi_1\phi_2}\ \cdots\ \overbracket{\cdots\ \phi_{2n}}\ . \tag{5.83}$$

That is, write down first all possible different patterns of contractions and then sum up them with the equal numerical coefficient, which is just +1.

Gell-Mann−Low Formula : The generating functional of Green's function (5.73) is expressed, using the notation (5.75), as

$$Z[J] = \langle e^{i\int d^4x(\mathcal{L}_{\rm int}(\phi)+J\phi)}\rangle_0\ . \tag{5.84}$$

The Green's function is given by

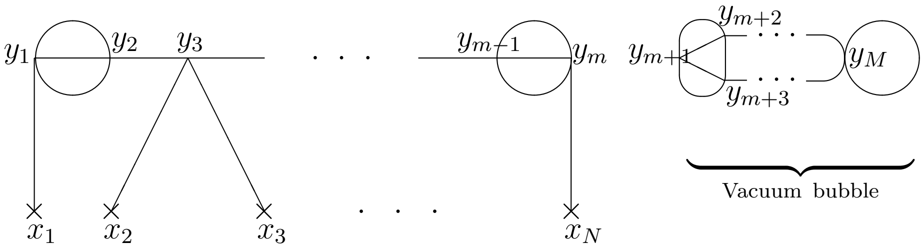
$$\begin{aligned} G(x_1,\cdots,x_N) &\equiv \langle 0|T[\hat\phi(x_1)\cdots\hat\phi(x_N)]|0\rangle \\ &= \frac{1}{Z[J]}\frac{1}{i^N}\frac{\delta^N Z[J]}{\delta J(x_1)\cdots\delta J(x_N)}\bigg|_{J=0} \\ &= \frac{\langle\phi(x_1)\cdots\phi(x_N)e^{i\int d^4y\mathcal{L}_{\rm int}(\phi)}\rangle_0}{\langle e^{i\int d^4y\mathcal{L}_{\rm int}(\phi)}\rangle_0}\ . \end{aligned} \tag{5.85}$$

Let us first investigate the numerator, using the notation $\phi_i \equiv \phi(x_i)$;

$$\langle\phi_1\cdots\phi_N e^{i\int d^4y\mathcal{L}_{\rm int}(\phi(y))}\rangle_0 = \sum_{M=0}^{\infty}\frac{i^M}{M!}\int d^4y_1\cdots d^4y_M\ \langle\phi_1\cdots\phi_N\mathcal{L}_{\rm int}(\phi(y_1))\cdots\mathcal{L}_{\rm int}(\phi(y_M))\rangle_0\ . \tag{5.86}$$

We call the points $x_1,\cdots,x_N$ the external points, and points $y_1,\cdots,y_M$ the interaction points. The contraction line attached to x_i is called external line, and contraction line which connects two interaction points y_i and y_j is called internal line.

When we apply the Wick's theorem and take the contraction of the right-hand-side of (5.86), we realize that, the interaction points y_i are classified to two categories, say, $\mathcal{C}$ and $\mathcal{V}$. The interaction points in category $\mathcal{C}$ are those, which can be traced, through the successive contraction lines, to at least one of the external points x_i. The interaction points in category $\mathcal{V}$ are those, which are connected to none of the external points, and the contractions are closed within the interaction points in category $\mathcal{V}$. Graphically, it is expressed as follows :



The above example contains m interaction points $y_1, \cdots, y_m$ in category $\mathcal{C}$ and $M-m$ interaction points $y_{m+1}, \cdots, y_M$ in category $\mathcal{V}$. The graph, which consists of interaction points in category $\mathcal{V}$ is called vacuum bubble. Let us express the collection of this type of amplitudes as

$$\langle\langle\phi_1 \cdots \phi_N \mathcal{L}_{\rm int}(y_1) \cdots \mathcal{L}_{\rm int}(y_m)\rangle\rangle_0 \times \langle\mathcal{L}_{\rm int}(y_{m+1}) \cdots \mathcal{L}_{\rm int}(y_M)\rangle_0 \ . \tag{5.87}$$

The second factor is the vacuum bubble. The simbol $\langle\langle\cdots\rangle\rangle_0$ in the first factor stands for the contractions which do not contain vacuum bubble.

Notice that the indices i $(i = 1, \cdots, M)$ of y_i are dummy and they are integrated by $\int d^4y_i$ in (5.86). Therefore, the amplitude (5.87) must be multiplied by the factor $\frac{M!}{m!(M-m)!}$, which counts the multiplicity of dividing M interaction points to m points in category $\mathcal{C}$ and $M-m$ points in category $\mathcal{V}$. Thus, (5.86) is written as

$$\begin{aligned}
\langle\phi_1 \cdots \phi_N e^{i\int d^4y \mathcal{L}_{\rm int}(\phi(y))}\rangle_0 &= \sum_{M=0}^{\infty} \frac{i^M}{M!} \int d^4y_1 \cdots d^4y_M \sum_{m=0}^{M} \langle\langle\phi_1 \cdots \phi_N \mathcal{L}_{\rm int}(y_1) \cdots \mathcal{L}_{\rm int}(y_m)\rangle\rangle_0 \\
&\quad \times \langle\mathcal{L}_{\rm int}(y_{m+1}) \cdots \mathcal{L}_{\rm int}(y_M)\rangle_0 \frac{M!}{m!(M-m)!} \\
&= \sum_{m=0}^{\infty} \frac{i^m}{m!} \int d^4y_1 \cdots d^4y_m \langle\langle\phi_1 \cdots \phi_N \mathcal{L}_{\rm int}(y_1) \cdots \mathcal{L}_{\rm int}(y_m)\rangle\rangle_0 \\
&\quad \times \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4y_{m+1} \cdots d^4y_{m+n} \langle\mathcal{L}_{\rm int}(y_{m+1}) \cdots \mathcal{L}_{\rm int}(y_{m+n})\rangle_0 \\
&= \langle\langle\phi_1 \cdots \phi_N e^{i\int d^4y \mathcal{L}_{\rm int}}\rangle\rangle_0 \langle e^{i\int d^4y \mathcal{L}_{\rm int}}\rangle_0 \ ,
\end{aligned} \tag{5.88}$$

where we have changed the sum over M by the sum over $n \equiv M - m$. When the numerator of (5.85) is replaced by this expression, the vacuum bubble amplitude $\langle e^{i\int d^4y \mathcal{L}_{\rm int}}\rangle_0$ is precisely cancelled by the denominator, and we arrive at a very simple result

$$G(x_1, \cdots, x_N) = \langle\langle\phi_1 \cdots \phi_N e^{i\int d^4y \mathcal{L}_{\rm int}}\rangle\rangle_0 \ . \tag{5.89}$$

The Green's function is obtained by simply removing, in the Wick's expansion of $\phi_1 \cdots \phi_N e^{i\int d^4y \mathcal{L}_{\rm int}}$, the graphs which contain vacuum bubbles. This is called Gell-Mann–Low formula.

Problem (5.3-1) Consider the self-interacting real scalar field described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{5.90}$$

The N-point Green's function is

$$G(x_1, \cdots, x_N) = \langle\langle\phi_1 \cdots \phi_N e^{-\frac{i\lambda}{4!}\int d^4y \phi^4}\rangle\rangle_0 \ . \tag{5.91}$$

Let us expand $G(x_1, \cdots, x_N)$ in the perturbation series as $G(\cdots) = \sum_{n=0}^{\infty}(-i\lambda)^n G^{(n)}(\cdots)$. Then, we have

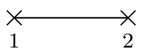
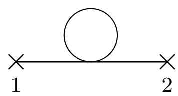
$$
\begin{aligned}
G^{(0)}(\cdots) &= \langle\langle \phi_1 \cdots \phi_N \rangle\rangle_0 \ , \\
G^{(1)}(\cdots) &= \langle\langle \phi_1 \cdots \phi_N \frac{1}{4!}\int d^4y \phi_y^4 \rangle\rangle_0 \ , \\
G^{(2)}(\cdots) &= \langle\langle \phi_1 \cdots \phi_N \frac{1}{2}\frac{1}{4!}\int d^4z \phi_z^4 \frac{1}{4!}\int d^4y \phi_y^4 \rangle\rangle_0 \ , \\
& \cdot \quad \cdot \quad \cdot \ .
\end{aligned}
\tag{5.92}
$$

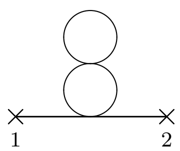
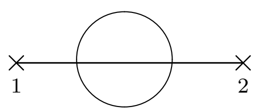
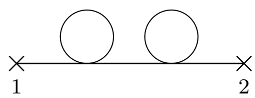
Derive the following results for the 2-point Green's function $G(x_1, x_2)$;

$$
G^{(0)}(x_1, x_2) = \phi_1\phi_2 \ , \quad G^{(1)}(x_1, x_2) = \int d^4y \frac{1}{2}\ \phi_1\phi_y\phi_y\phi_y\phi_y\phi_2
\tag{5.93}
$$

$$
\begin{aligned}
G^{(2)}(x_1, x_2) = \int d^4y \int d^4z \Bigg(& \frac{1}{2}\frac{1}{2!}\phi_1\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_2 \\
& + \frac{1}{3!}\phi_1\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_2 + \frac{1}{2}\frac{1}{2}\phi_1\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_2 \Bigg) \ ,
\end{aligned}
\tag{5.94}
$$

where $\phi_j\phi_k \equiv i\Delta_F(x_j - x_k)$ is the contraction. In the graphical representation, they are

$G(x_1, x_2) \ =$  $+(-i\lambda)\frac{1}{2}$ 

$+ \ (-i\lambda)^2 \Big[\frac{1}{2}\frac{1}{2!}$  $+ \frac{1}{3!}$  $+ \frac{1}{2}\frac{1}{2}$  $\Big]$

$+ \ O(\lambda^3) \ .$ (5.95)

The 4-point Green's function $G(x_1, x_2, x_3, x_4)$ contains the terms which consist of the product of two 2-point Green's functions. Extracting them explicitly, let us write

$$
\begin{aligned}
G(x_1, x_2, x_3, x_4) &= G(x_1, x_2)G(x_3, x_4) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3) \\
&+ \ G_c(x_1, x_2, x_3, x_4) \ .
\end{aligned}
\tag{5.96}
$$

The function $G_c(x_1, x_2, x_3, x_4)$ represents the sum of graphs in which all external points $x_1 \sim x_4$ are connected by propagators. It is called connected Green's function. Check their following perturbation expansions ;

$$G_c^{(1)}(x_1,x_2,x_3,x_4) = \int d^4y \; \phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_3\phi_4 \; , \tag{5.97}$$

$$\begin{aligned} G_c^{(2)}(x_1,x_2,x_3,x_4) = \int d^4y \int d^4z \frac{1}{2} \, (\; & \phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_3\phi_4 \\ & +\phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_3\phi_4 \\ & +\phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_3\phi_4 \\ & +\phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_3\phi_4 \\ & +\phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_3\phi_4 \\ & +\phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_3\phi_4 \\ & +\phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z\phi_z\phi_3\phi_4 \;) \; . \end{aligned} \tag{5.98}$$

Also, represent these amplitudes graphically following (5.95).

5.4 Path Integral of Fermi Field

Anticommutator of Fermi Field : The anticommutators of fermi fields are

$$\{\hat\psi_i(\mathbf{x}), \hat\pi_{\psi_j}(\mathbf{y})\} = i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \; , \; \{\hat\psi_i(\mathbf{x}), \hat\psi_j(\mathbf{y})\} = \{\hat\pi_{\psi_i}(\mathbf{x}), \hat\pi_{\psi_j}(\mathbf{y})\} = 0 \; . \tag{5.99}$$

According to the discretization method of the space by lattice $\mathbf{x} \to \boldsymbol{\alpha}$ given in §2.1, we discretize the fields using the notation $\alpha \equiv (i, \boldsymbol{\alpha})$ as

$$\hat\psi_i(\mathbf{x}) \to \hat\psi_\alpha \; , \; \hat\pi_{\psi_i}(\mathbf{x}) = i\hat\psi_i^\dagger(\mathbf{x}) \to \hat\pi_\alpha \equiv \frac{1}{\epsilon^3}\hat p_\alpha \; . \tag{5.100}$$

The anticommutators are then

$$\{\hat\psi_\alpha, \hat p_\beta\} = i\delta_{\alpha\beta} \; , \; \{\hat\psi_\alpha, \hat\psi_\beta\} = \{\hat p_\alpha, \hat p_\beta\} = 0 \; , \qquad \delta_{\alpha\beta} \equiv \delta_{ij}\delta^3_{\boldsymbol{\alpha\beta}} \; . \tag{5.101}$$

Eigenstate of $\hat{\boldsymbol\psi}_{\boldsymbol\alpha}, \hat{\boldsymbol p}_{\boldsymbol\alpha}$: The peculiar feature of fermi system is the existence of the state $|\underline{0}\rangle$ which satisfies, for all α, the condition

$$\hat\psi_\alpha|\underline{0}\rangle = 0 \; . \tag{5.102}$$

When this does not vanish, redefine $|\underline{0}\rangle$ by $\hat\psi_\alpha|\underline{0}\rangle$. Owing to the anticommutator, $\hat\psi_\alpha\hat\psi_\alpha = 0$, this state satisfies (5.102). Don't confuse $|\underline{0}\rangle$ with the vacuum $|0\rangle$.

Since $\hat p_\alpha \propto \hat\psi_\alpha^\dagger$, we also have

$$\langle\underline{0}|\hat p_\alpha = 0 \; . \tag{5.103}$$

We normalize $|\underline{0}\rangle$ by

$$\langle \underline{0}|\underline{0}\rangle = 1 \ . \tag{5.104}$$

Now, we define the eigenstates of $\hat{\psi}_\alpha$ and $\hat{p}_\alpha$ by

$$\hat{\psi}_\alpha|\psi\rangle = \psi_\alpha|\psi\rangle \ , \quad \langle p|\hat{p}_\alpha = \langle p|p_\alpha \ . \tag{5.105}$$

Since the operators $\hat{\psi}_\alpha$ and $\hat{p}_\alpha$ are subject to the anticommutators, their eigenvalues cannot be ordinary numbers. They must be anticommuting "numbers" called Grassmann numbers, satisfying

$$\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha \ , \quad p_\alpha p_\beta = -p_\beta p_\alpha \ , \quad \psi_\alpha p_\beta = -p_\beta\psi_\alpha \ , \quad \psi_\alpha\psi_\alpha = p_\alpha p_\alpha = 0 \ . \tag{5.106}$$

They also anticommute with operators $\hat{\psi}_\alpha$ and $\hat{p}_\alpha$.

Let us introduce a notation

$$\psi \cdot \hat{p} \equiv \sum_\alpha \psi_\alpha \hat{p}_\alpha \ , \tag{5.107}$$

and calculate $e^{-i\psi\cdot\hat{p}}\hat{\psi}_\alpha e^{i\psi\cdot\hat{p}}$ using the formula $e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \cdots$. The second term is

$$[-i\psi\cdot\hat{p}, \hat{\psi}_\alpha] = -i\sum_\beta [\psi_\beta\hat{p}_\beta, \hat{\psi}_\alpha] = -i\sum_\beta \psi_\beta\{\hat{p}_\beta, \hat{\psi}_\alpha\} = \psi_\alpha \ . \tag{5.108}$$

Since $[-i\psi\cdot\hat{p}, \psi_\alpha] = 0$, the third and the higher terms vanish, and we have

$$e^{-i\psi\cdot\hat{p}}\hat{\psi}_\alpha e^{i\psi\cdot\hat{p}} = \hat{\psi}_\alpha + \psi_\alpha \quad \text{or equivalently} \quad \hat{\psi}_\alpha e^{i\psi\cdot\hat{p}} = e^{i\psi\cdot\hat{p}}\hat{\psi}_\alpha + \psi_\alpha e^{i\psi\cdot\hat{p}} \ . \tag{5.109}$$

Operating the second expression to $|\underline{0}\rangle$, we obtain, owing to $\hat{\psi}_\alpha|\underline{0}\rangle = 0$,

$$\hat{\psi}_\alpha e^{i\psi\cdot\hat{p}}|\underline{0}\rangle = \psi_\alpha e^{i\psi\cdot\hat{p}}|\underline{0}\rangle \ . \tag{5.110}$$

Thus, the eigenstate $|\psi\rangle$ is expressed as

$$|\psi\rangle = e^{i\psi\cdot\hat{p}}|\underline{0}\rangle \ . \tag{5.111}$$

Similarly, from

$$e^{-i\psi\cdot\hat{p}}\hat{p}_\alpha e^{i\psi\cdot\hat{p}} = \hat{p}_\alpha + p_\alpha \ , \tag{5.112}$$

we obtain

$$\langle p| = \langle \underline{0}|e^{-ip\cdot\hat{\psi}} \ . \tag{5.113}$$

The bracket of $\langle p|$ and $|\psi\rangle$ is

$$\langle p|\psi\rangle = \langle \underline{0}|e^{-ip\cdot\hat{\psi}}|\psi\rangle = e^{-ip\cdot\psi}\langle \underline{0}|\psi\rangle = e^{-ip\cdot\psi}\langle \underline{0}|e^{-i\psi\cdot\hat{p}}|\underline{0}\rangle = e^{-ip\cdot\psi}\langle \underline{0}|\underline{0}\rangle = e^{-ip\cdot\psi} \ . \tag{5.114}$$

Rules of Derivative and Integral for Grassmann Number : Due to the special nature of Grassmann number, $\psi_\alpha\psi_\alpha = 0$, $p_\alpha p_\alpha = 0$, the general term, which consists of the product of many ψ's and p's, contains ψ_α or p_α with specific α at most once. Therefore, it is sufficient to define the derivative of Grassmann number in the way

$$\frac{\delta}{\delta\psi_\alpha}\ \psi_\alpha = 1 \ , \ \frac{\delta}{\delta\psi_\alpha}\ 1 = 0 \ , \ \frac{\delta}{\delta p_\alpha}\ p_\alpha = 1 \ , \ \frac{\delta}{\delta p_\alpha}\ 1 = 0 \ . \tag{5.115}$$

We must, however, be careful in the ordering of the Grasmann numbers in the product. We follow a convention called left-derivative. For example,

$$\frac{\delta}{\delta\psi_\alpha}(\psi_\alpha p_\beta) = p_\beta \ , \ \frac{\delta}{\delta\psi_\alpha}(p_\beta\psi_\alpha) = \frac{\delta}{\delta\psi_\alpha}(-\psi_\alpha p_\beta) = -p_\beta \ . \tag{5.116}$$

That is, the Grassmann differential $\delta/\delta\psi_\alpha$ stands for the procedure to take off ψ_α from the product "after" shifting it to the head of the term using anticommutation. To be consistent with the anticommutation of Grassmann number, the differentials must be subject to the anticommutations;

$$\frac{\delta}{\delta\psi_\alpha}\frac{\delta}{\delta\psi_\beta} = -\frac{\delta}{\delta\psi_\beta}\frac{\delta}{\delta\psi_\alpha} \ , \quad \frac{\delta}{\delta p_\alpha}\frac{\delta}{\delta p_\beta} = -\frac{\delta}{\delta p_\beta}\frac{\delta}{\delta p_\alpha} \ , \quad \frac{\delta}{\delta\psi_\alpha}\frac{\delta}{\delta p_\beta} = -\frac{\delta}{\delta p_\beta}\frac{\delta}{\delta\psi_\alpha} \ . \tag{5.117}$$

They must also anticommute with the fermionic operators $\hat{\psi}_\alpha,\ \hat{p}_\alpha$.

Let us proceed to the integration over Grassmann number, $\int d\psi_\alpha$. In defining it, we take care of the conformity with the definition of the differential $\delta/\delta\psi_\alpha$. As in the case of bosonic integral, we require that the partial-integration formula

$$\int d\psi_\alpha \frac{\delta}{\delta\psi_\alpha} f(\psi, p) = 0 \tag{5.118}$$

holds for any function $f(\psi, p)$. Since the integrand does not contain ψ_α, this formula implies

$$\int d\psi_\alpha\ 1 = 0 \qquad \text{and similarly} \qquad \int dp_\alpha\ 1 = 0 \ . \tag{5.119}$$

The remaining integration formulas, in the convenient normalization convention, are

$$\int d\psi_\alpha\ \psi_\alpha = 1 \ , \qquad \int dp_\alpha\ p_\alpha = 1 \ . \tag{5.120}$$

Comparing these results with (5.115), we find that the integral is equivalent to the differential. As in the case of differential, we must be careful of the ordering. For example,

$$\int d\psi_\alpha dp_\beta \psi_\alpha p_\beta = \int d\psi_\alpha dp_\beta(-p_\beta\psi_\alpha) = \int d\psi_\alpha(-\psi_\alpha) = -1 \ . \tag{5.121}$$

This implies that $d\psi_\alpha$ and dp_α are also subject to the anticommutation ;

$$d\psi_\alpha d\psi_\beta = -d\psi_\beta d\psi_\alpha \ , \quad dp_\alpha dp_\beta = -dp_\beta dp_\alpha \ , \quad d\psi_\alpha dp_\beta = -dp_\beta d\psi_\alpha \ . \tag{5.122}$$

They also anticommute with $\hat{\psi}_\alpha,\ \hat{p}_\alpha$.

Completeness Equation : In order to formulate the path integral for fermionic fields, we need, as in the case of bosonic case, the completeness equation. In the bosonic case, combining the two completeness equations, we have

$$\int dqdp|q\rangle\langle q|p\rangle\langle p| = \int dqdp|q\rangle \frac{1}{\sqrt{2\pi}} e^{ipq}\langle p| = \hat{1} \ . \tag{5.123}$$

Corresponding to this equation, let us first define

$$\hat{E}(\psi, p) \equiv |\psi\rangle e^{ip\cdot\psi}\langle p| \ . \tag{5.124}$$

According to the definition of $|\psi\rangle$, the action of $\hat{p}_\alpha$ on $|\psi\rangle$ is represented as

$$\hat{p}_\alpha|\psi\rangle = \hat{p}_\alpha e^{i\psi\cdot\hat{p}}|\underline{0}\rangle = \frac{\delta}{i\delta\psi_\alpha} e^{i\psi\cdot\hat{p}}|\underline{0}\rangle = \frac{\delta}{i\delta\psi_\alpha}|\psi\rangle \ . \tag{5.125}$$

Similarly, we have

$$\langle p|\hat{\psi}_\alpha = \frac{\delta}{-i\delta p_\alpha}\langle p| \ . \tag{5.126}$$

From these relations, we obtain

$$\begin{aligned}
\hat{E}(\psi,p)\hat{p}_\alpha &= p_\alpha \hat{E}(\psi,p) \ , &(5.127)\\
\hat{p}_\alpha \hat{E}(\psi,p) &= \left(\frac{\delta}{i\delta\psi_\alpha}|\psi\rangle\right) e^{ip\cdot\psi}\langle p| \\
&= \frac{\delta}{i\delta\psi_\alpha}\hat{E}(\psi,p) - |\psi\rangle\left(\frac{\delta}{i\delta\psi_\alpha}e^{ip\cdot\psi}\langle p|\right) \\
&= \frac{\delta}{i\delta\psi_\alpha}\hat{E}(\psi,p) + p_\alpha\hat{E}(\psi,p) \ . &(5.128)
\end{aligned}$$

Therefore, $\hat{E}(\psi,p)$ satisfies the differential equation

$$[\hat{p}_\alpha, \hat{E}(\psi,p)] = \frac{\delta}{i\delta\psi_\alpha}\hat{E}(\psi,p) \ . \tag{5.129}$$

Similar calculation for $\hat{\psi}_\alpha$ gives

$$[\hat{\psi}_\alpha, \hat{E}(\psi,p)] = \frac{\delta}{i\delta p_\alpha}\hat{E}(\psi,p) \ . \tag{5.130}$$

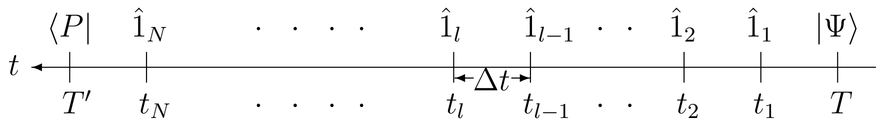
When (5.129) and (5.130) are integrated over all ψ_α and p_α, the right-hand-sides vanish due to (5.118). Thus, the operator $\int\prod_\alpha d\psi_\alpha dp_\alpha \hat{E}(\psi,p)$ commutes with all $\hat{\psi}_\alpha$ and $\hat{p}_\alpha$. Consequently, it must be proportional to the unit operator $\hat{1}$. The overall constant is fixed by the requirement $\hat{1}|\underline{0}\rangle = |\underline{0}\rangle$. The result is

$$\int\prod_\alpha\left(\frac{d\psi_\alpha dp_\alpha}{i}\right)|\psi\rangle e^{ip\cdot\psi}\langle p| = \hat{1} \ . \tag{5.131}$$

Transition Amplitude : Let us define the Hamiltonian $\hat{H}(\hat{p},\hat{\psi})$ so that $\hat{p}$ stands the left of $\hat{\psi}$, as in the case of scalar field. The transition amplitude from an eigenstate $|\Psi\rangle$, $\hat{\psi}_\alpha|\Psi\rangle = \Psi_\alpha|\Psi\rangle$, at an initial time $t = T$ to an eigenstate $\langle P|$, $\langle P|\hat{p}_\alpha = \langle P|P_\alpha$, at a final time $t = T'$ is given by

$$\langle P, T'|\Psi, T\rangle = \langle P|e^{-i(T'-T)\hat{H}}|\Psi\rangle \ . \tag{5.132}$$

We divide the time interval $T' - T$ by N intermediate times t_l $(l = 1, \cdots, N)$ with the equal time spacing Δt, and insert at each time t_l the completeness equations (5.131) adding a suffix l.



Then, the transition amplitude (5.132) is decomposed to the sequential product of subamplitudes of the form

$$e^{ip_l\cdot\psi_l}\langle p_l|e^{-i\Delta t\hat{H}(\hat{p},\hat{\psi})}|\psi_{l-1}\rangle = e^{ip_l\cdot\psi_l}e^{-i\Delta t H(p_l,\psi_{l-1})}e^{-ip_l\cdot\psi_{l-1}} = e^{i\Delta t\left[\frac{p_l\cdot(\psi_l-\psi_{l-1})}{\Delta t} - H(p_l,\psi_{l-1})\right]} \ . \tag{5.133}$$

Remembering the relation $p_\alpha = \epsilon^3\pi_\alpha$ and defining $\pi_\alpha \equiv \bar{\psi}_\alpha\gamma^0$, we realize that the exponent is expressed, in the continuum limit of the 3-dimensional space, as

$$[\cdots] = \int d^3\mathbf{x}\ \pi_i(\mathbf{x},t)\dot{\psi}_i(\mathbf{x},t) - H(t) = L(t) = \int d^3\mathbf{x}\ \mathcal{L}(\mathbf{x},t) \ . \tag{5.134}$$

Thus, the transition amplitude is expressed, up to an uninteresting overall constant factor, as

$$\langle P, T'|\Psi, T\rangle = \int [d\psi d\bar{\psi}] e^{i\int_T^{T'} dt \int d^3\mathbf{x}\ \mathcal{L}\left(\bar{\psi}(\mathbf{x},t),\psi(\mathbf{x},t)\right)} \ . \tag{5.135}$$

Green's Function and Generating Functional : Green's function is expressed by the path integral as

$$\begin{aligned}\langle 0|T[\hat{\psi}_i(x_1)\cdots\hat{\bar{\psi}}_j(y_1)\cdots]|0\rangle &\equiv G(\psi_i(x_1)\cdots\bar{\psi}_j(y_1)\cdots) \\ &= \frac{\int[d\psi d\bar{\psi}]\psi_i(x_1)\cdots\bar{\psi}_j(y_1)\cdots e^{i\int d^4x\mathcal{L}(x)}}{\int[d\psi d\bar{\psi}]e^{i\int d^4x\mathcal{L}(x)}} \ .\end{aligned} \tag{5.136}$$

Introducing the anticommuting Grassmann sources $\eta(x)$ and $\bar{\eta}(x)$, we define the generating functional by

$$Z[\eta,\bar{\eta}] = \int [d\psi d\bar{\psi}] e^{i\int d^4x(\mathcal{L}(x)+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x))} \ . \tag{5.137}$$

Then, the Green function is obtained by

$$G(\psi_i(x_1)\cdots\bar{\psi}_j(y_1)\cdots) = \frac{1}{Z[\eta,\bar{\eta}]}\frac{\delta}{i\delta\bar{\eta}_i(x_1)}\cdots\frac{\delta}{-i\delta\eta_j(y_1)}\cdots Z[\eta,\bar{\eta}]\bigg|_{\eta=\bar{\eta}=0} \ . \tag{5.138}$$

We must be careful that $\frac{\delta}{\delta\eta_i}$ and $\frac{\delta}{\delta\bar{\eta}_i}$ are anticommuting ;

$$\frac{\delta}{\delta\eta_1}\frac{\delta}{\delta\eta_2} = -\frac{\delta}{\delta\eta_2}\frac{\delta}{\delta\eta_1} \ , \ \cdots \ . \tag{5.139}$$

Some Remarks on Grassmann Integral

1. Let us first consider the case of single Grassmann variable ψ. Any function of ψ has a form

$$f(\psi) = f_0 + \psi f_1 \ . \tag{5.140}$$

Now, we shift the variable from ψ to $\psi' \equiv \psi + \eta$ with the Grassmann number η. Since,

$$f(\psi') = f(\psi+\eta) = f_0 + (\psi+\eta)f_1 = f_0 + \eta f_1 + \psi f_1 \ , \tag{5.141}$$

we have

$$\int d\psi' f(\psi') = f_1 \ , \quad \int d\psi f(\psi') = \int d\psi f(\psi+\eta) = f_1 \ . \tag{5.142}$$

Thus, we realize, under the shift of the Grassmann variable,

$$\psi' = \psi + \eta \implies d\psi' = d\psi \ . \tag{5.143}$$

In the case of many Grassmann variables ψ_α $(\alpha = 1,\cdots,N)$, we also have

$$\psi'_\alpha = \psi_\alpha + \eta_\alpha \implies [d\psi'] = [d\psi] \equiv d\psi_1\cdots d\psi_N \ . \tag{5.144}$$

2. Let us next consider the linear transformation using the ordinary number $A_{\beta\alpha}$;

$$\psi'_\beta = \sum_{\alpha=1}^{N} A_{\beta\alpha}\psi_\alpha \qquad : \beta = 1,\cdots,N \ . \tag{5.145}$$

The N-th product of ψ' is

$$\psi'_1 \cdots \psi'_N = \sum_{\{\alpha_i\}} A_{1\alpha_1} \cdots A_{N\alpha_N} \psi_{\alpha_1} \cdots \psi_{\alpha_N} = (\det A)\psi_1 \cdots \psi_N \ . \qquad (5.146)$$

From the identity

$$\int [d\psi]\psi_1 \cdots \psi_N = \int [d\psi']\psi'_1 \cdots \psi'_N = \int [d\psi'](\det A)\psi_1 \cdots \psi_N \ , \qquad (5.147)$$

we find a slightly surprising result

$$\psi'_\beta = \sum_{\alpha=1}^{N} A_{\beta\alpha}\psi_\alpha \implies [d\psi'] = (\det A)^{-1}[d\psi] \ . \qquad (5.148)$$

3. Let us consider the following integral ;

$$\int [d\psi d\bar{\chi}] e^{\sum_{\alpha\beta} \bar{\chi}_\alpha M_{\alpha\beta}\psi_\beta} \ . \qquad (5.149)$$

ψ's and $\bar{\chi}$'s are Grassmann variables and $M_{\alpha\beta}$ are ordinary numbers. If we define $\psi' = M\psi$, then $[d\psi] = \det M[d\psi']$ and we obtain

$$\int [d\psi d\bar{\chi}] e^{\sum_{\alpha\beta} \bar{\chi}_\alpha M_{\alpha\beta}\psi_\beta} = \det M \int [d\psi' d\bar{\chi}] e^{\sum_\alpha \bar{\chi}_\alpha \psi'_\alpha} \propto \det M \ . \qquad (5.150)$$

This should be compared with the bosonic case, where

$$\phi' = M\phi \implies [d\phi'] = (\det M)[d\phi] \ , \qquad (5.151)$$

and

$$\int [d\phi d\pi] e^{\sum_{\alpha\beta} \pi_\alpha M_{\alpha\beta}\phi_\beta} \propto (\det M)^{-1} \ . \qquad (5.152)$$

Generating Functional for Free Dirac Field : The Lagrangian for the free Dirac field is

$$\mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi \ , \qquad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \ . \qquad (5.153)$$

So, the exponent of (5.137) is, inserting the $i\epsilon$ term as in the case of scalar field,

$$\begin{aligned} &\int d^4x(\bar{\psi}(i\partial\!\!\!/ - m + i\epsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta) \\ &= \int d^4x d^4y \left\{ [\bar{\psi} + \bar{\eta}(i\partial\!\!\!/ - m + i\epsilon)^{-1}]_x (i\partial\!\!\!/ - m + i\epsilon)_{xy} [\psi + (i\partial\!\!\!/ - m + i\epsilon)^{-1}\eta]_y \right. \\ &\qquad \left. -\bar{\eta}_x (i\partial\!\!\!/ - m + i\epsilon)^{-1}_{xy} \eta_y \right\} \ . \end{aligned} \qquad (5.154)$$

Taking a similar process as that of scalar field given by (5.60) to (5.62), and shifting the integral variables, we obtain the generating functional for free Dirac field ;

$$\begin{aligned} Z_0[\eta, \bar{\eta}] &\equiv \int [d\psi d\bar{\psi}] e^{i\int d^4x(\mathcal{L}_0 + \bar{\eta}\psi + \bar{\psi}\eta)} \\ &\propto e^{-i\int d^4x d^4y \ \bar{\eta}(x)(\partial\!\!\!/ - m + i\epsilon)^{-1}_{xy}\eta(y)} \\ &= e^{-i\int d^4x d^4y \ \bar{\eta}(x) S_F(x-y)\eta(y)} \ , \end{aligned} \qquad (5.155)$$

where $S_F(x-y)$ is just a Feynman propagator for Dirac field,

$$S_F(x-y) = (i\partial\!\!\!/ - m + i\epsilon)^{-1}\delta^4(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k\!\!\!/ - m + i\epsilon}e^{-ik\cdot(x-y)} \ . \tag{5.156}$$

Fermion-Boson Interacting System : As an example of interactions of fermions and bosons, let us consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} \ , \tag{5.157}$$

$$\mathcal{L}_0 = \bar\psi(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2) \ , \tag{5.158}$$

$$\mathcal{L}_{\rm int}(\phi,\bar\psi,\psi) = -g\phi\bar\psi\psi - \frac{1}{4!}\lambda\phi^4 \ . \tag{5.159}$$

The generating functional is

$$\begin{aligned} Z[J,\eta,\bar\eta] &\equiv \int [d\phi d\psi d\psi^\dagger] e^{i\int d^4x[\mathcal{L}+J\phi+\bar\eta\psi+\bar\psi\eta]} \\ &= \exp\left[i\int d^4x \mathcal{L}_{\rm int}\left(\frac{\delta}{i\delta J(x)}, \frac{\delta}{-i\delta\eta(x)}, \frac{\delta}{i\delta\bar\eta(x)}\right)\right] Z_0[J,\eta,\bar\eta] \ , \qquad (5.160) \\ Z_0[J,\eta,\bar\eta] &= \exp\left[-\int d^4x d^4y \frac{1}{2}J(x)i\Delta_F(x-y)J(y) - \int d^4x d^4y \ \bar\eta(x) iS_F(x-y)\eta(y)\right] . \qquad (5.161) \end{aligned}$$

Exploiting the trick (5.72), here again, we obtain the expression

$$\begin{aligned} Z[J,\eta,\bar\eta] &= \exp\left[\frac{1}{2}\int d^4x \frac{\delta}{\delta\phi(x)} i\Delta_F(x-y)\frac{\delta}{\delta\phi(y)} + \int d^4x d^4y \ (iS_F(x-y))_{ij}\frac{\delta}{\delta\bar\psi_j(y)}\frac{\delta}{\delta\psi_i(x)}\right] \\ &\quad \times \exp\left[i\int d^4y \ (\mathcal{L}_{\rm int}(\phi,\bar\psi,\psi) + J\phi + \bar\eta\psi + \bar\psi\eta)\right]\Bigg|_{\phi=\psi=\bar\psi=0} \ . \qquad (5.162) \end{aligned}$$

The Green's function is obtained in terms of the Wick's expansion $\langle\cdots\rangle_0$ and $\langle\langle\cdots\rangle\rangle_0$ as

$$\begin{aligned} &\langle 0|T[\hat\phi(x_1)\cdots\hat\psi(y_1)\cdots\hat{\bar\psi}(z_1)\cdots]|0\rangle \\ &\quad = \frac{1}{Z[J,\eta,\bar\eta]}\frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta\bar\eta(y_1)}\cdots\frac{\delta}{-i\delta\eta(z_1)}\cdots Z[J,\eta,\bar\eta]\Bigg|_{J=\eta=\bar\eta=0} \\ &\quad = \frac{\langle\phi(x_1)\cdots\psi(y_1)\cdots\bar\psi(z_1)\cdots e^{i\int d^4x \ \mathcal{L}_{\rm int}(\phi,\bar\psi,\psi)}\rangle_0}{\langle e^{i\int d^4x \ \mathcal{L}_{\rm int}(\phi,\bar\psi,\psi)}\rangle_0} \\ &\quad = \langle\langle\phi(x_1)\cdots\psi(y_1)\cdots\bar\psi(z_1)\cdots e^{i\int d^4x \ \mathcal{L}_{\rm int}(\phi,\bar\psi,\psi)}\rangle\rangle_0 \ . \qquad (5.163) \end{aligned}$$

Some Remarks on Fermion Contraction

1. The contraction of fermi fields is defined as

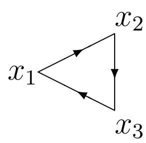
$$\int d^4x d^4y \ (iS_F(x-y))_{ij}\frac{\delta}{\delta\bar\psi_j(y)}\frac{\delta}{\delta\psi_i(x)} \ \psi_{i_1}(x_1)\bar\psi_{i_2}(x_2) = \overbrace{\psi_{i_1}(x_1)\bar\psi_{i_2}}(x_2) = (iS_F(x_1-x_2))_{i_1i_2} \ . \tag{5.164}$$

2. We must be careful, in the Wick's expansion, the minus sign resulting in the change of the ordering of ψ's and $\bar\psi$'s. For example,

$$\begin{aligned}\langle\psi_{i_1}(x_1)\bar{\psi}_{i_2}(x_2)\psi_{i_3}(x_3)\bar{\psi}_{i_4}(x_4)\rangle_0 &= \psi_1\bar{\psi}_2\psi_3\bar{\psi}_4 + \psi_1\bar{\psi}_2\psi_3\bar{\psi}_4 = \psi_1\bar{\psi}_2\psi_3\bar{\psi}_4 - \psi_1\bar{\psi}_4\psi_3\bar{\psi}_2 \\ &= (iS_F(x_1-x_2))_{i_1i_2}(iS_F(x_3-x_4))_{i_3i_4} - (iS_F(x_1-x_4))_{i_1i_4}(iS_F(x_3-x_2))_{i_3i_2} \quad (5.165)\end{aligned}$$

3. Each fermion loop gives the minus sign to the amplitude. For example,

$$(\bar{\psi}(x_1)\psi(x_1))(\bar{\psi}(x_2)\psi(x_2))(\bar{\psi}(x_3)\psi(x_3)) =$$



$$= -\mathrm{tr}\left[iS_F(x_1-x_2)iS_F(x_2-x_3)iS_F(x_3-x_1)\right] \; . \qquad (5.166)$$

Problem (5.4-1) Let us consider the fermion system, $\{\hat{\psi}_\alpha, \hat{p}_\beta\} = i\delta_{\alpha\beta}$. The exponential of the Grassmann nubmer, $e^{ip\cdot\psi}$, when Taylor expanded, terminates due to $\psi_\alpha\psi_\alpha = 0$. Check the expression $e^{ip\cdot\psi} = \prod_\alpha(1 + ip_\alpha\psi_\alpha)$. Using this expression, show the completeness equation

$$\hat{1} \equiv \int \prod_\alpha \left(\frac{d\psi_\alpha dp_\alpha}{i}\right)|\psi\rangle e^{ip\cdot\psi}\langle p| \qquad (5.167)$$

$$|\psi\rangle \equiv e^{i\psi\cdot\hat{p}}|\underline{0}\rangle \; , \quad \langle p| \equiv \langle\underline{0}|e^{-ip\cdot\hat{\psi}} \; , \quad \hat{\psi}_\alpha|\underline{0}\rangle = 0 \; , \quad \langle\underline{0}|\hat{p}_\alpha = 0 \; , \quad \langle\underline{0}|\underline{0}\rangle = 1 \qquad (5.168)$$

$$\int d\psi_\alpha\psi_\alpha = \int dp_\alpha p_\alpha = 1 \; , \quad \int d\psi_\alpha \; 1 = \int dp_\alpha \; 1 = 0 \; , \qquad (5.169)$$

really realizes

$$\hat{1}|\underline{0}\rangle = |\underline{0}\rangle \; . \qquad (5.170)$$

Let us next consider the product $\hat{1}\hat{1}$.

$$\begin{aligned}\hat{1}\hat{1} &\equiv \int\prod_\alpha\left(\frac{d\psi_\alpha dp_\alpha}{i}\right)|\psi\rangle e^{ip\cdot\psi}\langle p| \times \int\prod_\alpha\left(\frac{d\psi'_\alpha dp'_\alpha}{i}\right)|\psi'\rangle e^{ip'\cdot\psi'}\langle p'| \\ &= \int\prod_\alpha\left(\frac{d\psi_\alpha dp'_\alpha}{i}\right)|\psi\rangle\int\prod_\alpha\left(\frac{dp_\alpha d\psi'_\alpha}{i}\right)e^{ip\cdot\psi}\langle p|\psi'\rangle e^{ip'\cdot\psi'}\langle p'| \; . \qquad (5.171)\end{aligned}$$

Substitute $\langle p|\psi'\rangle = e^{-ip\cdot\psi'}$ and expand the product of three exponentials. By integrating over $dp_\alpha d\psi'_\alpha$, prove the relation

$$\hat{1}\hat{1} = \hat{1} \; . \qquad (5.172)$$

Problem (5.4-2) Let us consider the Lagrangian of fermi field $\psi(x)$ with four-fermi interaction;

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{G}{2}\bar{\psi}\psi\bar{\psi}\psi \; . \qquad (5.173)$$

Let us expand the two point Green's function

$$G(x_1,x_2)_{j_1j_2} \equiv \langle 0|T[\psi_{j_1}(x_1)\bar{\psi}_{j_2}(x_2)]|0\rangle = \langle\langle\psi_{j_1}(x_1)\bar{\psi}_{j_2}(x_2)e^{-\frac{iG}{2}\int d^4y\bar{\psi}\psi\bar{\psi}\psi}\rangle\rangle_0 \qquad (5.174)$$

by perturbation expansion as $G(x_1,x_2)_{j_1j_2} = \sum_{n=0}^{\infty}(-iG)^n G^{(n)}(x_1,x_2)_{j_1j_2}$. Check

$$G^{(0)}(x_1,x_2)_{j_1j_2} = (\psi_1\bar{\psi}_2)_{j_1j_2} \tag{5.175}$$

$$G^{(1)}(x_1,x_2)_{j_1j_2} = \int d^4y \left(-\psi_1\bar{\psi}_y\psi_y\bar{\psi}_2 \mathrm{tr}(\psi_y\bar{\psi}_y) + \psi_1\bar{\psi}_y\psi_y\bar{\psi}_y\psi_y\bar{\psi}_2\right)_{j_1j_2} , \tag{5.176}$$

where $(\psi_j\bar{\psi}_k)_{j_jj_k} \equiv i(S_F(x_j - x_k))_{j_jj_k}$.

It is also recommended to derive the expression for $G^{(2)}(x_1,x_2)_{j_1j_2}$ It consists of ten terms with coefficient $+1$ and -1 for the terms which contain even number of tr and odd number of tr, respectively.

Problem (5.4-3) Let us consider the boson-fermion system with Yukawa interaction described by the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\phi\bar{\psi}\psi \ . \tag{5.177}$$

Show that the connected component of three point Green's function of ϕ

$$G(x_1,x_2,x_3) = \langle\langle \phi_1\phi_2\phi_3 e^{-ig\int d^4y\phi\bar{\psi}\psi}\rangle\rangle_0 \tag{5.178}$$

is expressed as

$$G_c(x_1,x_2,x_3) = -(-ig)^3 \int d^4x \int d^4y \int d^4z \phi_1\phi_x\phi_2\phi_y\phi_3\phi_z \times \mathrm{tr}\left(\psi_x\bar{\psi}_y\psi_y\bar{\psi}_z\psi_z\bar{\psi}_x + \psi_x\bar{\psi}_z\psi_z\bar{\psi}_y\psi_y\bar{\psi}_x\right) + O(g^5) \ . \tag{5.179}$$

Also, show the connected component of the four point Green's function of ϕ and ψ

$$\langle 0|T[\hat{\phi}(x_4)\hat{\psi}_k(x_3)\hat{\phi}(x_2)\hat{\bar{\psi}}_j(x_1)]|0\rangle = \langle\langle \phi_4\psi_{k3}\phi_2\bar{\psi}_{j1} e^{-ig\int d^4y\phi\bar{\psi}\psi}\rangle\rangle_0 \tag{5.180}$$

is expressed as

$$G_c(\phi_4\psi_{k3}\phi_2\bar{\psi}_{j1}) = (-ig)^2 \int d^4y \int d^4z \left(\phi_4\phi_y(\psi_3\bar{\psi}_y\psi_y\bar{\psi}_z\psi_z\bar{\psi}_1)_{kj}\phi_z\phi_2 + \phi_4\phi_y(\psi_3\bar{\psi}_z\psi_z\bar{\psi}_y\psi_y\bar{\psi}_1)_{kj}\phi_z\phi_2\right) + O(g^4) \ . \tag{5.181}$$

Give the graphical representation of these expressions.

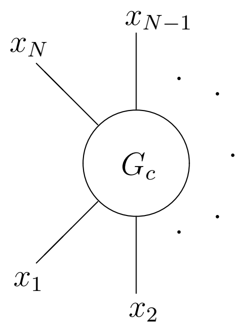
Chapter 6

Connected Green's Function and Effective Action

6.1 Connected Green's Function

The Green's function $G(x_1, \cdots, x_N)$ in general consists of the sum of many amplitudes corresponding to the various patterns of the Wick's contraction. The connected Green's function $G_c(x_1, \cdots, x_N)$ is defined as the limited sum of specific amplitudes, in which any two external points x_i and x_j are connected through the propagators :

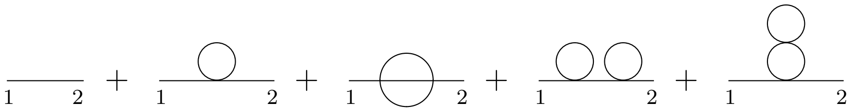
$$G_c(x_1, \cdots, x_N) =$$



$$\equiv \langle 0|T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)]|0\rangle_c \ . \tag{6.1}$$

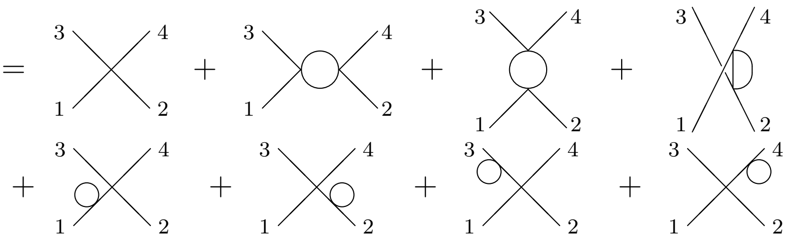
For example, in the $\lambda\phi^4$ theory, abbreviating the coefficients,

$$G_c(x_1, x_2) =$$



$$+ \cdots, \tag{6.2}$$

$$G_c(x_1, x_2, x_3, x_4) =$$



$$+ \cdots . \tag{6.3}$$

Let us define the generating functional $W[J]$ of G_c as well as $Z[J]$ of G by

$$Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!}\, i^N \!\int d^4x_1 \cdots d^4x_N \ J(x_1)\cdots J(x_N) G(x_1, \cdots, x_N) \ , \tag{6.4}$$

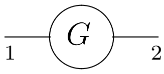
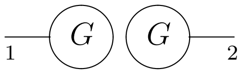
$$iW[J] = \sum_{N=0}^{\infty} \frac{1}{N!}\, i^N \!\int d^4x_1 \cdots d^4x_N \ J(x_1)\cdots J(x_N) G_c(x_1, \cdots, x_N) \ . \tag{6.5}$$

There is a remarkable relation between $Z[J]$ and $W[J]$. They are related by

$$Z[J] = e^{iW[J]} \; . \tag{6.6}$$

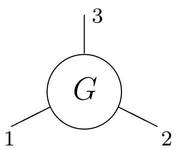
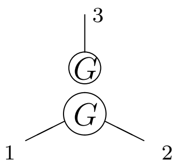
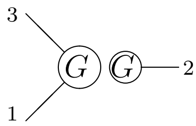
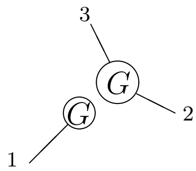
We will be convinced of this relation by the explicit calculations. For two-point function, we have

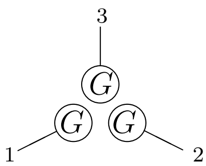
$$\begin{aligned} G_c(x_1, x_2) &= \left.\frac{\delta^2 iW[J]}{i\delta J_1 i\delta J_2}\right|_{J=0} = \left.\frac{\delta^2}{i\delta J_1 i\delta J_2} \ln Z[J]\right|_{J=0} = \left.\frac{\delta}{i\delta J_1}\left(\frac{1}{Z}\frac{\delta Z}{i\delta J_2}\right)\right|_0 \\ &= \left.\frac{1}{Z}\frac{\delta^2 Z}{i\delta J_1 i\delta J_2}\right|_0 - \left.\frac{1}{Z}\frac{\delta Z}{i\delta J_1}\right|_0 \left.\frac{1}{Z}\frac{\delta Z}{i\delta J_2}\right|_0 \equiv \left.\frac{Z_{12}}{Z}\right|_0 - \left.\frac{Z_1}{Z}\right|_0 \left.\frac{Z_2}{Z}\right|_0 \end{aligned}$$

$=$  $-$  . (6.7)

The second term just subtracts the disconected part of $G(x_1, x_2)$. For the three-point function, we have

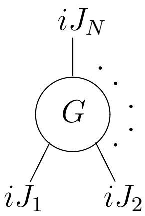
$$\begin{aligned} G_c(x_1, x_2, x_3) &= \left.\frac{\delta}{i\delta J_3}\left(\frac{1}{Z}Z_{12} - \frac{1}{Z}Z_1\frac{1}{Z}Z_2\right)\right|_0 \\ &= \left.\frac{Z_{123}}{Z}\right|_0 - \left.\frac{Z_{12}}{Z}\right|_0 \left.\frac{Z_3}{Z}\right|_0 - \left.\frac{Z_{13}}{Z}\right|_0 \left.\frac{Z_2}{Z}\right|_0 - \left.\frac{Z_{23}}{Z}\right|_0 \left.\frac{Z_1}{Z}\right|_0 + 2\left.\frac{Z_1}{Z}\right|_0 \left.\frac{Z_2}{Z}\right|_0 \left.\frac{Z_3}{Z}\right|_0 \end{aligned}$$

$=$  $-$  $-$  $-$ 

$+ 2 \times$  . (6.8)

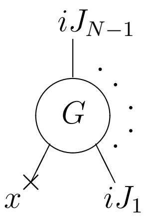
The final term is compensating the triple over-subtractions of the completely disconected graph.

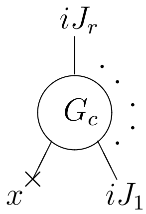
The proof is as follows. Let us symbolically express, abbreviating the 4-dimensional integrals, as

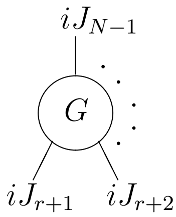
$Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!}$ 

. (6.9)

Take the derivative of $Z[J]$ with respect to $J(x)$. Then, we have

$\frac{\delta Z}{i\delta J(x)} = \sum_{N=1}^{\infty} \frac{1}{(N-1)!}$ 

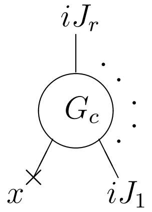
$= \sum_{N=1}^{\infty} \frac{1}{(N-1)!} \sum_{r=0}^{N-1} \frac{(N-1)!}{r!(N-1-r)!}$ 

$\times$ 

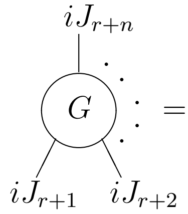
. (6.10)

In the second expression, the factor $\sum_{r=0}^{N-1} \frac{(N-1)!}{r!(N-1-r)!}$ comes from the multiplicity of the cases where, r-source terms out of $N-1$ source terms $iJ_1 \sim iJ_{N-1}$ are connected to the point x. Since the index of J is dummy, this coefficient is just the binomial coefficient. Rearranging the summation index N to $n \equiv N-1-r$, we have

$$\frac{\delta Z}{i\delta J(x)} = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{r!}$$



$$\times \frac{1}{n!}$$



$$= \frac{\delta iW[J]}{i\delta J(x)} Z[J] \ . \qquad (6.11)$$

The solution of this differential equation is (6.6).

6.2 Effective Action

The generating functional of Green's function is

$$Z[J] = \int [d\phi] e^{i\int d^4x(\mathcal{L}(\phi)+J\phi)} \ . \qquad (6.12)$$

This expression can be interpreted as the path integral of the quantum system which has interaction with the source $J(x)$. Under this interpretation, the "vacuum expectation value" of $\hat{\phi}(x)$ is

$$\begin{aligned}\varphi(x) &\equiv \langle 0|\hat{\phi}(x)|0\rangle_J = \frac{1}{Z[J]}\int [d\phi]\phi(x) e^{i\int d^4x(\mathcal{L}(\phi)+J\phi)} \\ &= \frac{1}{Z[J]}\frac{\delta Z[J]}{i\delta J(x)} = \frac{\delta}{i\delta J(x)} \ln Z[J] = \frac{\delta W[J]}{\delta J[J]} \ . \qquad (6.13)\end{aligned}$$

$\varphi(x) = \frac{\delta W[J]}{\delta J(x)}$ is a functional of J ; $\varphi = \varphi[J]$. $\varphi[J=0]$ gives the true vacuum expectation value of $\hat{\phi}(x)$. By solving $\varphi = \varphi[J]$ with respect to J, J may be given as an inverse functional ; $J(x) = J[\varphi]$.

Free Field Example : For a rough understanding, we give a result for the free field :

$$\mathcal{L} = \mathcal{L}_0(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \ , \qquad (6.14)$$

$$Z_0[J] = e^{-\frac{1}{2}\int d^4x d^4y J(x) i\Delta_F(x-y)J(y)} \ , \qquad (6.15)$$

$$W_0[J] = \frac{i}{2}\int d^4x d^4y \ J(x) i\Delta_F(x-y)J(y) \ , \qquad (6.16)$$

$$\varphi(x) = \frac{\delta W_0[J]}{\delta J(x)} = -\int d^4y \ \Delta_F(x-y)J(y) \ , \qquad (6.17)$$

$$J(x) = (\Box + m^2)\varphi(x) \quad \text{because} \quad (\Box + m^2)\Delta_F(x-y) = -\delta^4(x-y) \ . \qquad (6.18)$$

Effective Action : The Legendre transformation of $W[J]$ is called effective action $\Gamma[\varphi]$:

$$\Gamma[\varphi] = W[J] - \int d^4y \ J(y)\varphi(y) \ . \qquad (6.19)$$

In this equation, J is understood to be the functional of φ ;

$$J = J[\varphi] \quad \text{from} \quad \varphi(x) = \frac{\delta W[J]}{\delta J(x)} \ . \qquad (6.20)$$

Taking the functional derivative of $\Gamma[\varphi]$ with respect to $\varphi(x)$, we obtain

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \int d^4y \, \frac{\delta J(y)}{\delta\varphi(x)} \frac{\delta W[J]}{\delta J(y)} - \int d^4y \, \frac{\delta J(y)}{\delta\varphi(x)} \varphi(y) - J(x) \ . \tag{6.21}$$

The first and second terms cancel due to (6.20), and we find

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x) \ . \tag{6.22}$$

Free Field Example : We again examine the free field example :

$$\begin{aligned} \Gamma_0[\varphi] &= \frac{i}{2}\int d^4x d^4y \ J(x) i\Delta_F(x-y) J(y) - \int d^4y \ J(y)\varphi(y) \\ &= -\frac{1}{2}\int d^4y \ J(y)\varphi(y) \end{aligned} \tag{6.23}$$

Substituting $J(x) = (\Box + m^2)\varphi(x)$ and using the partial integration, we obtain

$$\begin{aligned} \Gamma_0[\varphi] &= -\frac{1}{2}\int d^4x \ \varphi(x)(\Box + m^2)\varphi(x) \\ &= \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 \right) \\ &= \int d^4x \ \mathcal{L}_0(\varphi) \ . \end{aligned} \tag{6.24}$$

This is nothing but a classical action integral.

When we obtained the effective action $\Gamma[\varphi]$ for the interacting field system, the vacuum expectation value of the field $\varphi[J=0]$ is determined by solving the equation

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = 0 \ . \tag{6.25}$$

This is a quantum mechanical "action principle" which contains all quantum effects. Thus, $\Gamma[\varphi]$ is called effective action.

Two-Point Function : From the relations $\frac{\delta\Gamma[\varphi]}{\delta\varphi} = -J(x)$ and $\frac{\delta W[J]}{\delta J(x)} = \varphi(x)$, we have

$$\Gamma(x,y) \equiv \frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = -\frac{\delta J(x)}{\delta\varphi(y)} \ , \tag{6.26}$$

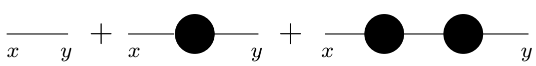
$$W(x,y) \equiv \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = \frac{\delta\varphi(x)}{\delta J(y)} \ . \tag{6.27}$$

Thus,

$$\int d^4z \ W(x,z)\Gamma(z,y) = -\int d^4z \frac{\delta\varphi(x)}{\delta J(z)}\frac{\delta J(z)}{\delta\varphi(y)} = -\frac{\delta\varphi(x)}{\delta\varphi(y)} = -\delta^4(x-y) \ , \tag{6.28}$$

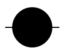
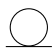
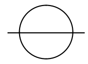
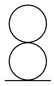
that is, $W(x,z)$ and $-\Gamma(z,y)$ are mutually in the inverse relation.

Let us take a limit $J = 0$ for a while. Then, $W(x,y)$ is a connected two-point Green's function, $W(x,y) = iG_c(x,y)$. The connected two-point Green's function is graphically expressed as

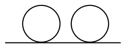
$G_c(x,y) =$  $+ \cdots ,$ (6.29) where the line is the propagator

$$\underset{x\quad y}{\rule{2em}{0.4pt}} = i\Delta_F(x-y) = i(-\Box - m^2 + i\epsilon)^{-1}_{xy} \ , \tag{6.30}$$

and the black circle is the collection of all graphs which cannot be separated to two graphs by cutting a single internal line in the graph ;

 $= i\Sigma =$  $+$  $+$  $+ \cdots .$ (6.31)

They are called "one-particle irreducible" graphs. If we cut the middle line of the graph

 , (6.32)

this is separated to mutually disconnected two graphs. Such a graph is called "one-particle redusible", and is not contained in the black circle $i\Sigma$ so that the series (6.29) does not double-count the same graph.

The series (6.29) is summed up, using $D \equiv -\Box - m^2 + i\epsilon$, as

$$\begin{aligned} G_c(x,y) &= iD^{-1} + iD^{-1}i\Sigma iD^{-1} + iD^{-1}i\Sigma iD^{-1}i\Sigma iD^{-1} + \cdots \\ &= iD^{-1}(1 - \Sigma D^{-1} + (\Sigma D^{-1})^2 - (\Sigma D^{-1})^3 + \cdots) \\ &= iD^{-1}(1+\Sigma D^{-1})^{-1} = i\left[(1+\Sigma D^{-1})D\right]^{-1} = i(D+\Sigma)^{-1} \\ &= i(-\Box - m^2 + \Sigma + i\epsilon)^{-1} \ . \end{aligned} \tag{6.33}$$

Therefore, we have

$$\Gamma(x,y) = iG_c^{-1}(x,y) = (-\Box - m^2 + \Sigma + i\epsilon)_{xy} = (-\Box - m^2 + i\epsilon)\delta^4(x-y) + \Sigma(x,y) \ . \tag{6.34}$$

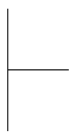
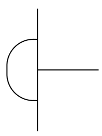
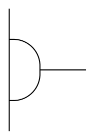
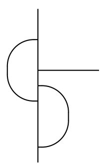
Let us express $\Gamma(x,y)$ as a Fourier integral,

$$\Gamma(x,y) = \int \frac{d^4k}{(2\pi)^4}(k^2 - m^2 + \Sigma(k^2) + i\epsilon)e^{-ik\cdot(x-y)} \ . \tag{6.35}$$

Then, the connected two-point Green's function is expressed as

$$G_c(x,y) = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + \Sigma(k^2) + i\epsilon}e^{-k\cdot(x-y)} \ . \tag{6.36}$$

Proper Vertex Function : In order to understand the effective action $\Gamma[\varphi]$ more deeply, let us define the, so-called, N-point proper vertex function $\Gamma(x_1, \cdots, x_N)$. $i\Gamma(x_1, \cdots, x_N)$ is defined from the N-point connected Green function $G_c(x_1, \cdots, x_N)$, by getting rid of the external propagators $i\Delta_F(x_1 - x_2)$ and removing the graphs which contain the one-particle reducible internal lines. For example, $i\Gamma(x_1, x_2, x_3)$ is expressed as a sum of series of graphs

 $+$  $+$  $+$  $+ \cdots .$ (6.37)

The surprising fact of the effective action $\Gamma[\varphi]$ is that, it is the generating functional of the proper vertex functions. That is,

$$\Gamma[\varphi] = \sum_{N=0}^{\infty}\frac{1}{N!}\int d^4x_1 \cdots d^4x_N \ \varphi(x_1)\cdots\varphi(x_N)\Gamma(x_1, \cdots, x_N) \ , \tag{6.38}$$

$$\Gamma(x_1, \cdots, x_N) = \left.\frac{\delta^N\Gamma[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_N)}\right|_{\varphi=0} \ . \tag{6.39}$$

From now on, we do not set $|_{\varphi=0}$, because we do not know whether $\varphi = 0$ is the true vacuum expectation value or not, and only keep the relation between $J(x)$ and $\varphi(x)$.

Expansion of $\boldsymbol{W[J]}$ in Terms of $\boldsymbol{\Gamma[\varphi]}$: We do not give, here, the rigorous proof of the above fact, and show some examples, which will be convincing enough.

Let us introduce the notation

$$W_{ijk\cdots} \equiv \frac{\delta^N W[J]}{\delta J_i \delta J_j \delta J_k \cdots} \ , \qquad J_i \equiv J(x_i) \tag{6.40}$$

$$\Gamma_{ijk\cdots} \equiv \frac{\delta^N \Gamma[\varphi]}{\delta\varphi_i \delta\varphi_j \delta\varphi_k \cdots} \ , \qquad \varphi_i \equiv \varphi(x_i) \ . \tag{6.41}$$

From the relations $\frac{\delta W}{\delta J_i} = \varphi_i$ and $\frac{\delta \Gamma}{\delta \varphi_i} = -J_i$, two point functions are

$$W_{ij} = \frac{\delta^2 W}{\delta J_i \delta J_j} = \frac{\delta \varphi_j}{\delta J_i} \ , \tag{6.42}$$

$$\Gamma_{ij} = \frac{\delta^2 \Gamma}{\delta\varphi_i \delta\varphi_J} = -\frac{\delta J_j}{\delta \varphi_i} \ . \tag{6.43}$$

Thus,

$$W_{ij}\Gamma_{jk} = -\frac{\delta\varphi_j}{\delta J_i}\frac{\delta J_k}{\delta \varphi_j} = -\frac{\delta J_k}{\delta J_i} = -\delta_{ik} \ , \tag{6.44}$$

where, in the first and second expressions, the integration $\int d^4x_j$ is tacitly assumed. According to the formula of partial differentiation, we have

$$\frac{\delta}{\delta J_i} = \frac{\delta\varphi_j}{\delta J_i}\frac{\delta}{\delta\varphi_j} = W_{ij}\frac{\delta}{\delta\varphi_j} \ . \tag{6.45}$$

Operation of this differential to $\Gamma_{ij\cdots}$ is

$$\frac{\delta}{\delta J_3}\Gamma_{ij\cdots} = W_{3k}\Gamma_{kij\cdots} \ . \tag{6.46}$$

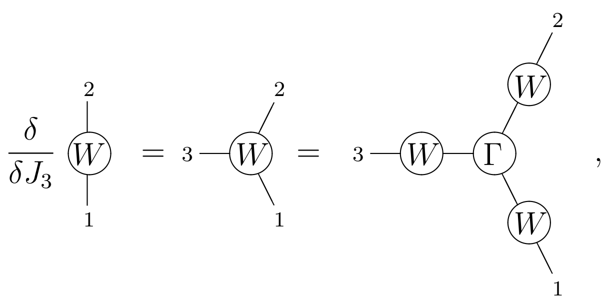
Let us operate this differential on (6.44). Since the right-hand-side vanishes, we have

$$0 = \frac{\delta}{\delta J_3}(W_{1i}\Gamma_{i2}) = W_{13i}\Gamma_{i2} + W_{1i}W_{3j}\Gamma_{ij2} \ . \tag{6.47}$$

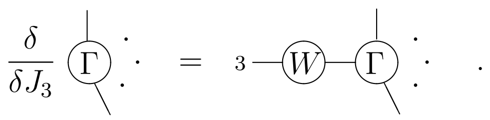
Using (6.44) once again, we obtain

$$\frac{\delta}{\delta J_3}W_{12} \equiv W_{123} = W_{1i}W_{2j}W_{3k}\Gamma_{ijk} \ . \tag{6.48}$$

The graphical representations of (6.48) and (6.46) are

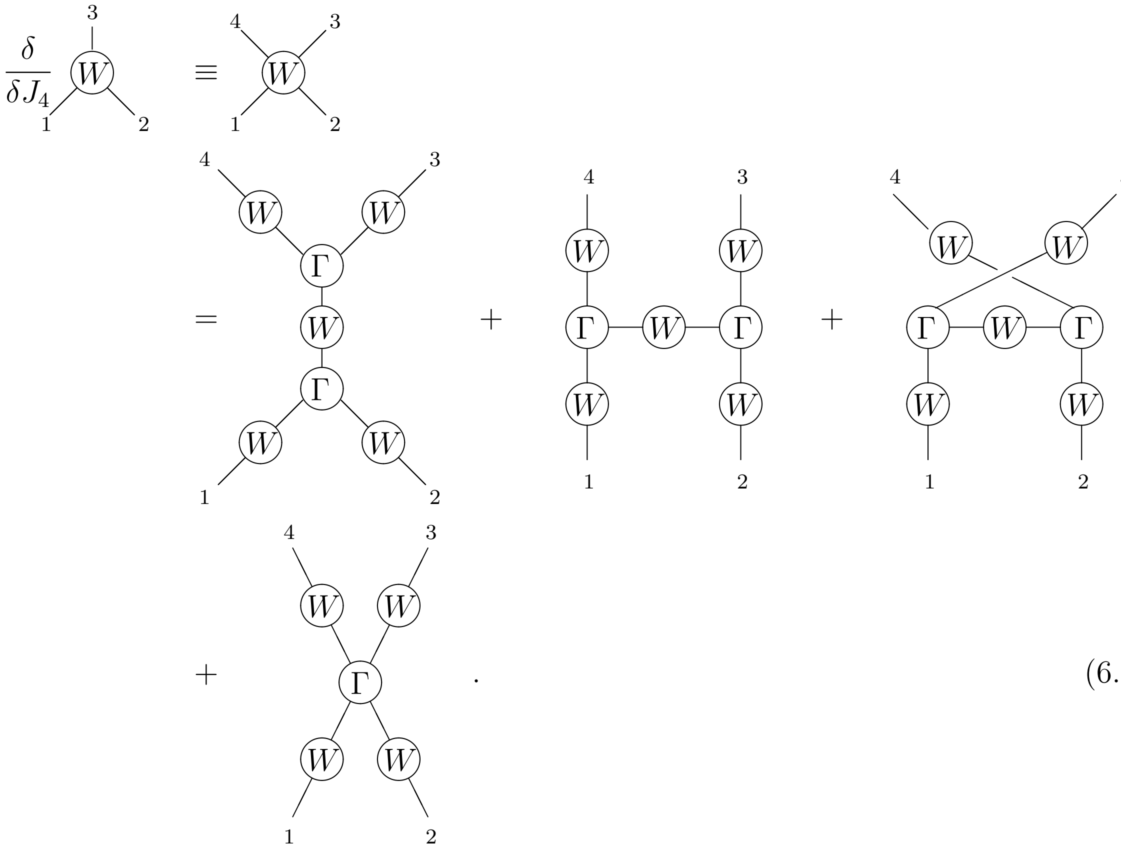


(6.49)



(6.50)

The four-point connected Green's function is obtained by differentiating (6.49) ;



(6.51)

These graphs exhaust all of the possible patterns of "trees" which represent the possible ways of the connection of the four external points $x_1 \sim x_4$. By iterating the same procedure, we can go to the N-point function with $N \geq 5$, and find the similar trees. If Γ were not one-particle irreducible, we would encounter the double counting of the same graph.

Since $W(x, y)$ is the inverse of $-\Gamma(x, y)$, all connected Green's functions are expressed as a sum of trees which are expressed in terms of Γ. That is, the effective action $\Gamma[\varphi]$ contains the all informations of the dynamics of the quantum field theory.

Chapter 7

Scattering Amplitude and Scattering Cross Section

7.1 Transition Amplitude

The one of the main subjects of the quantum field theory is to describe the scattering process of particles. The scatterings occur through the interactions between particles. Therefore, the probability amplitude of the scattering is related to the connected Green's function. Let us remember the form of the connected N-point Green's function

$$G_c(x_1, \cdots, x_N) = \langle 0|T[\hat{\phi}(x_1) \cdots \hat{\phi}(x_N)]|0\rangle_c \ . \tag{7.1}$$

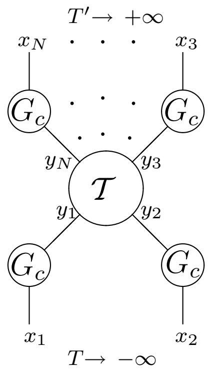
As we found in the previous chapter, every external points $x_1, \cdots, x_N$ of the Green's function are always attached by the connected two-point Green's function $G_c(x_i, y_i) = -iW(x_i, y_i)|_0$. Therefore, G_c is expressed as

$$G_c(x_1, \cdots, x_N) = \int \prod_{i=1}^{N} \left[d^4y_i \ G_c(x_i, y_i)\right] \mathcal{T}(y_1, \cdots, y_N) \ . \tag{7.2}$$

Now, we take a limit

$$x_1^0 = x_2^0 \equiv T \to -\infty \ , \quad x_3^0 = \cdots = x_N^0 \equiv T' \to +\infty \ . \tag{7.3}$$

Graphically, it is expressed as



. (7.4)

This Green's function expresses the probability amplitude of the "physical reaction" in which, particles 1 and 2 are created at the space-time points x_1 and x_2 at the initial time $T \to -\infty$, and as the time develops, they begin to interact and particles $3 \sim N$ come out from the interaction region, and they are annihirated at $x_3 \sim x_N$ at the final time $T' \to \infty$.

Let us inspect the role of the external two-point functions $G_c(x_i, y_i)$. In the limit $T \to -\infty$, $G_c(x_1, y_1)$ takes a form

$$G_c(x_1, y_1) \longrightarrow \lim_{x_1^0 = T \to -\infty} \langle 0|T[\hat{\phi}(x_1)\hat{\phi}(y_1)]|0\rangle_c = \langle 0|\hat{\phi}(y_1) \lim_{T \to -\infty} \hat{\phi}(x_1)|0\rangle_c \ . \tag{7.5}$$

The suffix c shows that this function does not contain the disconnected part $\langle 0|\hat{\phi}(y_1)|0\rangle\langle 0|\hat{\phi}(x_1)|0\rangle$. The state $\hat{\phi}(x_1)|0\rangle$ in the right-hand-side of (7.5) is the state we prepared for the particle 1 at the initial time $T \to -\infty$. The role of $\langle 0|\hat{\phi}(y_1)$ is the inspection that the particle surely exists as a one-particle state at space-time point y_1. This means that the right-hand-side of (7.5) is nothing but the wave function of particle 1. On the other hand, $G_c(x_3, y_3)$ in the limit $T' \to +\infty$ is

$$G_c(x_3, y_3) \longrightarrow \lim_{x_3^0 = T' \to +\infty} \langle 0|T[\hat{\phi}(x_3)\hat{\phi}(y_3)]|0\rangle_c = \langle 0|\hat{\phi}(y_3) \lim_{T' \to +\infty} \hat{\phi}(x_3)|0\rangle_c^* \ . \tag{7.6}$$

This is the complex conjugate of the wave function of the state $\hat{\phi}(x_3)|0\rangle$, which we prepare to detect the particle 3 at $T' \to \infty$.

In the "realistic" scattering processes, all particles, $1 \sim N$, both in initial state ($T \to -\infty$) and in final state ($T' \to +\infty$), are eigenstates of energy-momentum operators $\hat{P}^\mu \equiv (\hat{H}, \hat{\mathbf{P}})$;

$$\hat{P}^\mu |p_i\rangle = p_i^\mu |p_i\rangle \ , \qquad p_i^2 = m_i^2 \ , \qquad i = 1, 2, \cdots, N \ . \tag{7.7}$$

What we have to do for these reaction processes is to replace all $G_c(x_i, y_i)$ in (7.2) with the corresponding wave functions

$$\langle 0|\hat{\phi}(y_i)|p_i\rangle \equiv f_i(y_i) \quad \text{for} \quad i = 1, 2 \ , \tag{7.8}$$
$$\langle 0|\hat{\phi}(y_i)|p_i\rangle^* \equiv f_i^*(y_i) \quad \text{for} \quad i = 3, \cdots, N \ . \tag{7.9}$$

Thus, the transition amplitude from the initial state $|p_1\rangle|p_2\rangle$ at $T \to -\infty$ to the final state $|p_3\rangle \cdots |p_N\rangle$ at $T' \to +\infty$ is given by

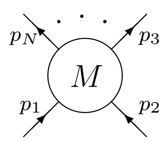
$$T(p_1, p_2 \ ; \ p_3, \cdots, p_N) = \int [\prod_i d^4 x_i] f_1(x_1) f_2(x_2) f_3^*(x_3) \cdots f_N^*(x_N) \ \mathcal{T}(x_1, x_2, x_3, \cdots, x_N) \tag{7.10}$$

Owing to the conservation of energy and momentum, the transition amplitude $T(p_i)$, in general, contains the factor

$$\delta^4(p_\alpha - p_\beta) \ , \qquad p_\alpha \equiv p_1 + p_2 \ , \qquad p_\beta \equiv p_3 + \cdots + p_N \ . \tag{7.11}$$

Extracting this factor and some conventional factors from $T(p_i)$, we define $M(p_i)$ by

$$T(p_1, p_2 \ ; \ p_3, \cdots, p_N) = i(2\pi)^4 \delta^4(p_\alpha - p_\beta) M(p_1, p_2 \ ; \ p_3, \cdots, p_N) \ . \tag{7.12}$$



(7.13)

The function $M(p_i)$ is called scattering amplitude.

7.2 Scattering Cross Section

Normalization of One-Particle State : We normalize the one-particle state $|p\rangle$ by so-called Lorentz covariant normalization ;

$$\langle p'|p\rangle = (2\pi)^3 2p^0 \delta^3(\mathbf{p}' - \mathbf{p}) \ . \tag{7.14}$$

Then, the completeness equation for the one-particle state which realizes $\hat{1}^{(1)}|p\rangle = |p\rangle$ is

$$\hat{1}^{(1)} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} |p\rangle\langle p| \ . \tag{7.15}$$

The norm of $|p\rangle$ is

$$\langle p|p\rangle = 2p^0(2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}) = 2p^0 \int d^3\mathbf{x}\ e^{i(\mathbf{p}-\mathbf{p})\cdot\mathbf{x}} = 2p^0 V\big|_{V\to\infty} \ , \tag{7.16}$$

where V is the total volume of the three-dimensional space. This normalization can be interpreted, for the initial particle's state $|p\rangle$, as if there are $2p^0$ particles per unit volume. That is, the number density of the particle is $\rho = 2p^0$. As we may understand from the fact that the charge density ρ_c is the time component j_c^0 of the 4-dimensional charge current j_c^μ, ρ and p^0 have a same Lorentz transformation property. This is the origin of the name of the normalization (7.14).

As for the final particle's state $\langle p|$, we have, for an arbitrary one-particle state $|\psi\rangle$,

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} |\langle p|\psi\rangle|^2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} \langle\psi|p\rangle\langle p|\psi\rangle = \langle\psi|\psi\rangle \ . \tag{7.17}$$

From this expression, we find that the factor $\frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}$ works as the weight for the probability to find the particle having a momentum in the vicinity of $\mathbf{p}$ within the $\mathbf{p}$-space volume $d^3\mathbf{p}$.

Transition "Probability"(Number) : The differential number of times of the transition, dN, is now expressed as

$$dN = \left|(2\pi)^4\delta^4(p_\alpha - p_\beta) M(p_1, p_2\ ;\ p_3, \cdots, p_N)\right|^2 \prod_{f=3}^{N} \frac{d^3\mathbf{p}_f}{(2\pi)^3 2p_f^0} \ . \tag{7.18}$$

The square of the delta function is

$$\begin{aligned} |(2\pi)^4\delta^4(p_\alpha - p_\beta)|^2 &= (2\pi)^4\delta^4(p_\alpha - p_\beta)(2\pi)^4\delta^4(p_\alpha - p_\alpha) \\ &= (2\pi)^4\delta^4(p_\alpha - p_\beta) \int d^4x\ e^{i(p_\alpha - p_\alpha)\cdot x} \\ &= (2\pi)^4\delta^4(p_\alpha - p_\beta)\ TV|_{T\to\infty, V\to\infty} \ , \end{aligned} \tag{7.19}$$

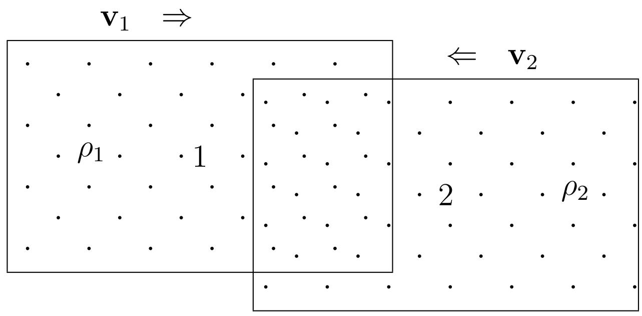
where T is the total time interval. Thus, we have

$$dN = TV(2\pi)^4\delta^4(p_\alpha - p_\beta)\,|M(p_1, p_2\ ;\ p_3, \cdots, p_N)|^2 \prod_{f=3}^{N} \frac{d^3\mathbf{p}_f}{(2\pi)^3 2p_f^0} \ . \tag{7.20}$$

Scattering Cross Section : We are considering the reaction

$$1\ +\ 2\ \longrightarrow\ 3\ +\ 4\ +\ \cdots\ +\ N\ . \tag{7.21}$$

Let us prepare the experimental setup. Suppose the particles of species 1 are uniformly distributed in the large box with the number density ρ_1, and moving with the common velocity $\mathbf{v}_1$ together with the motion of the box. The particles of species 2 are also distributed in the other box with the number density ρ_2 and common velosity $\mathbf{v}_2$. We assume, for simplicity, $\mathbf{v}_1$ and $\mathbf{v}_2$ are anti-parallel.



The relative velocity of the particle 1 and 2 is

$$v_{\text{rel}} = |\mathbf{v}_1 - \mathbf{v}_2| \ . \tag{7.22}$$

Let N be the total number of times the reaction (7.21) occurs within the time interval T in the volume V which is embedded in the overlapping domain of the two boxes. Evidently, N should be proportional to V, T, ρ_1, ρ_2 and v_{rel}. Thus,

$$N = VT\rho_1\rho_2 \ v_{\text{rel}} \ \sigma \ . \tag{7.23}$$

This is the experimental definition of the scattering cross section σ of the reaction (7.21). When any restriction is imposed on the states of the final particles $3 \sim N$, we write a differential quantity as

$$dN = VT\rho_1\rho_2 \ v_{\text{rel}} \ d\sigma \ . \tag{7.24}$$

This defines the differential cross section $d\sigma$. Comparing this expression with (7.20), and remembering the normalization of the initial particle's state $|p\rangle$, which states

$$\rho_1 = 2p_1^0 \ , \quad \rho_2 = 2p_2^0 \ , \tag{7.25}$$

we obtain the formula for the differential cross section ;

$$d\sigma = \frac{1}{4p_1^0 \ p_2^0 \ v_{\text{rel}}}(2\pi)^4\delta^4(p_\alpha - p_\beta)\,|M(p_1, p_2 \ ; \ p_3, \cdots, p_N)|^2 \prod_{f=3}^{N} \frac{d^3\mathbf{p}_f}{(2\pi)^3 2p_f^0} \ . \tag{7.26}$$

Remark : According to the formula (1.21), the velocity $\mathbf{v}$ of a particle is related to its 4-momentum p^μ by

$$\mathbf{v} = \frac{\mathbf{p}}{p^0} \ . \tag{7.27}$$

Therefore, the relative velocity v_{rel} is

$$v_{\text{rel}} = \left| \frac{\mathbf{p}_1}{p_1^0} - \frac{\mathbf{p}_2}{p_2^0} \right| \ . \tag{7.28}$$

Taking the square of this expression and comparing it with the Lorentz invariant quantity

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \ , \tag{7.29}$$

we find that $v_{\rm rel}$ is identical to the quantity

$$\bar{v}_{\rm rel} \equiv \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1^0 \ p_2^0} \ . \tag{7.30}$$

Although this equality is realized only when $\mathbf{p}_1$ and $\mathbf{p}_2$ are (anti-) parallel, the substitution of $v_{\rm rel}$ in (7.23), (7.24) and (7.26) with $\bar{v}_{\rm rel}$ has a great significance. Look at the equation (7.23), for example. Since the left-hand-side is the number of counting of the reaction, it should be independent of the inertial frame where the experiment is performed. The product VT on the right-hand-side is the Lorentz invaliant volume of the 4-dimensional space-time (the two factors from time delay and space contraction are canceled). The equations (7.25) and (7.30) insist that the product $\rho_1 \rho_2 \ \bar{v}_{\rm rel}$ is also Lorentz invariant. Thus the cross section defined by

$$N = VT \rho_1 \rho_2 \ \bar{v}_{\rm rel} \ \sigma \tag{7.31}$$

is a universal quantity which is independent of the inertial frame where the experiment is carried out.

Problem (7.2-1) Suppose the reaction is $1 + 2 \to 3 + 4$. Show that the differential cross section

$$d\sigma = \frac{1}{4 p_1^0 p_2^0 \bar{v}_{\rm rel}} (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) |M(p_i)|^2 \frac{d^3\mathbf{p}_3}{(2\pi)^3 2p_3^0} \frac{d^3\mathbf{p}_4}{(2\pi)^3 2p_4^0} \tag{7.32}$$

is expressed in the center-of-momentum frame ($\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}, \ \mathbf{p}_3 = -\mathbf{p}_4 \equiv \mathbf{p}'$) as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|\mathbf{p}'|}{(p_1^0 + p_2^0)^2 |\mathbf{p}|} |M(p_i)|^2 \ , \tag{7.33}$$

where $d\Omega$ is a solid angle of $d\mathbf{p}'$.

7.3 One-Particle Wave Function

The calculation of the scattering amplitude requires the knowledge of the one-particle wave function

$$f(x) = \langle 0 | \hat{\phi}(x) | p \rangle \ . \tag{7.34}$$

From the commutation relations of $\hat{\phi}$ with $\hat{H}$ and $\hat{\mathbf{P}}$, it is now straightforward to show

$$\hat{\phi}(x) = e^{i\hat{p}\cdot x} \hat{\phi}(0) e^{-i\hat{p}\cdot x} \ , \qquad \hat{p}^\mu \equiv (\hat{H}, \hat{\mathbf{P}}) \ . \tag{7.35}$$

Therefore, $f(x)$ is expressed as

$$f(x) = \langle 0 | e^{i\hat{p}\cdot x} \hat{\phi}(0) e^{-i\hat{p}\cdot x} | p \rangle = \langle 0 | \hat{\phi}(0) | p \rangle e^{-ip\cdot x} \ . \tag{7.36}$$

In the case of free scalar field, $|p\rangle$ and $\hat{\phi}(0)$ are given by

$$|p\rangle = \sqrt{(2\pi)^3 2p^0} \hat{a}_{\mathbf{p}}^\dagger |0\rangle \ , \qquad p^0 \equiv \omega_p = \sqrt{\mathbf{p}^2 + m^2} \ , \tag{7.37}$$

$$\hat{\phi}(0) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} (\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger) \ . \tag{7.38}$$

From these expressions, the coefficient of (7.36) is calculated to be

$$\langle 0|\hat{\phi}(0)|p\rangle = \langle 0| \int d^3\mathbf{k}\sqrt{\frac{\omega_p}{\omega_k}}\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = \int d^3\mathbf{k}\delta^3(\mathbf{k}-\mathbf{p}) = 1 \ . \tag{7.39}$$

Therefore, we have

$$f(x) = e^{-ip\cdot x} \ . \tag{7.40}$$

In the case of interacting field, $f(x)$ generally takes the form

$$f(x) = Z^{1/2}e^{-ip\cdot x} \ . \tag{7.41}$$

The coefficient $Z^{1/2}$ is a numerical factor depending on the interaction. The square of 4-momentum p^μ is also shifted from its free-field value m^2 to the physical mass-square

$$p^2 = m^2_{\text{phys}} \ , \tag{7.42}$$

which represents the experimentally measurable mass.

Determination of m_{phys} and Z : Let us return to the expression of two point Green's function;

$$\begin{aligned} G(x_1,x_2) &= \langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)]|0\rangle \\ &= \theta(x_1^0-x_2^0)\langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle + \theta(x_2^0-x_1^0)\langle 0|\hat{\phi}(x_2)\hat{\phi}(x_1)|0\rangle \ . \end{aligned} \tag{7.43}$$

We insert the identity operator $\hat{1}$ between the product of two $\hat{\phi}$'s. $\hat{1}$ represents the completeness of the state vectors of the quantum system. It is generally given as

$$\hat{1} = |0\rangle\langle 0| + \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}|p\rangle\langle p| + [\text{ multi-particle states }] \ . \tag{7.44}$$

The first term gives the disconnected part of $G(x_1,x_2)$, which is merely a product of $\langle 0|\hat{\phi}(x_1)|0\rangle$ and $\langle 0|\hat{\phi}(x_2)|0\rangle$. The important observation comes from the second term, $G^{(1)}(x_1,x_2)$, where the one-particle completeness equation is inserted to $G(x_1,x_2)$. From the definitions (7.34) and (7.41), we obtain

$$\begin{aligned} G^{(1)}(x_1,x_2) &\equiv \theta(x_1^0-x_2^0)\langle 0|\hat{\phi}(x_1)\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}|p\rangle\langle p|\hat{\phi}(x_2)|0\rangle + (1\leftrightarrow 2) \\ &= \theta(x_1^0-x_2^0)\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}e^{-ip\cdot(x_1-x_2)}Z + \theta(x_2^0-x_1^0)\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}e^{ip\cdot(x_1-x_2)}Z \end{aligned} \tag{7.45}$$

Recalling the procedure used in §2.4 for the free scalar field, we find this is just the free scalar propagator with mass m_{phys} with the additional factor Z ;

$$G^{(1)}(x_1,x_2) = i\int \frac{d^4p}{(2\pi)^4}\frac{Z}{p^2-m^2_{\text{phys}}+i\epsilon}e^{-ip\cdot(x_1-x_2)} \ . \tag{7.46}$$

From this result, we realize that the Fourier component of the two-point Green function (6.36)

$$G(x_1,x_2) = i\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m^2+\Sigma(p^2)+i\epsilon}e^{-ip\cdot(x_1-x_2)} \tag{7.47}$$

has a pole at $p^2 = m^2_{\text{phys}}$, and the residue of the pole determines the magnitude of Z.

The position of the pole is a value of p^2 at which the denominator of the Fourier component of (7.47) vanishes ;

$$m_{\rm phys}^2 - m^2 + \Sigma(m_{\rm phys}^2) = 0 \ . \tag{7.48}$$

The residue Z is determined by expanding the denominator around $p^2 = m_{\rm phys}^2$;

$$\begin{aligned} p^2 - m^2 + \Sigma(p^2) &= p^2 - m_{\rm phys}^2 + \Sigma(p^2) - \Sigma(m_{\rm phys}^2) \\ &= p^2 - m_{\rm phys}^2 + \Sigma'(m_{\rm phys}^2)(p^2 - m_{\rm phys}^2) + O\left((p^2 - m_{\rm phys}^2)^2\right) \\ &= \left(p^2 - m_{\rm phys}^2\right)\left(1 + \Sigma'(m_{\rm phys}^2)\right) + O\left((p^2 - m_{\rm phys}^2)^2\right) \ , \end{aligned} \tag{7.49}$$

where $\Sigma'(p^2) = d\Sigma(p^2)/dp^2$. Thus, Z is expressed as

$$Z = \frac{1}{1 + \Sigma'(m_{\rm phys}^2)} \ . \tag{7.50}$$

Remark : Suppose we are calculating the lowest-order scattering amplitude of some reaction process according to the perturbation theory. In this case, we are always allowed to use the free particle wave function $f(x)$ by setting $Z = 1$ and $m_{\rm phys} = m$, because the deviation from them belongs to the higher-order quantities, which we are neglecting.

When we calculate the scattering amplitudes to the higher orders, it is often convenient, from a practical point of view, to treat the mass parameter m, appearing in the original Lagrangian, as a function of $m_{\rm phys}$ by solving (7.48) perturbatively with respect to m, because the scattering amplitudes are now expressed in terms of the experimentally observable $m_{\rm phys}$.

Problem (7.3-1) Let us calculate the scattering amplitude of ϕ^4 theory whose lowest-order connected 4-point Green's function is given by (5.97). The transition amplitude $T(p_1, p_2; p_3, p_4)$ is obtained by replacing the external 2-point Green's functions by the wave functions of initial and final particles

$$f_1(y) = e^{-ip_1\cdot y} \ , \ f_2(y) = e^{-ip_2\cdot y} \ , \ f_3^*(y) = e^{ip_3\cdot y} \ , \ f_4^*(y) = e^{ip_4\cdot y} \ . \tag{7.51}$$

Check the result

$$\begin{aligned} T(p_1, p_2; p_3, p_4) &= -i\lambda \int d^4y f_1(y) f_2(y) f_3^*(y) f_4^*(y) = -i\lambda \int d^4y \ e^{i(p_3+p_4-p_1-p_2)\cdot y} \\ &= -i\lambda (2\pi)^4 \delta(p_3 + p_4 - p_1 - p_2) \ . \end{aligned} \tag{7.52}$$

Therefore, the scattering amplitude is $M = -\lambda$.

Problem (7.3-2) Let us calculate the lowest order scattering amplitude of ψ and ϕ interacting through Yukawa interaction. The 4-point connected Green's function is given by (5.181). What we have to do is to replace the external 2-point functions by the wave functions. Check the following replacement procedure

$$\overbracket{\psi_z\bar{\psi}_1} \to \langle 0|\hat{\psi}(z)|p_1, s\rangle = \ u_{\mathbf{p}_1,s} e^{-ip_1\cdot z} \ , \tag{7.53}$$

$$\overbracket{\psi_3\bar{\psi}_y} \to \langle p_3, s'|\hat{\bar{\psi}}(y)|0\rangle = \ \bar{u}_{\mathbf{p}_3,s'} e^{ip_3\cdot y} \ , \tag{7.54}$$

$$\overbracket{\phi_z\phi_2} \to f_2(z) = e^{-ip_2\cdot z} \ , \tag{7.55}$$

$$\overbracket{\phi_4\phi_y} \to f_4^*(y) = e^{ip_4\cdot y} \ , \tag{7.56}$$

and calculate the scattering amplitude. The result is given by

$$M(p_1, s, p_2; p_3, s', p_4) = (-ig)^2 \ \bar{u}_{\mathbf{p}_3,s'} \left(\frac{1}{\not{p}_1 + \not{p}_2 - m + i\epsilon} + \frac{1}{\not{p}_1 - \not{p}_4 - m + i\epsilon} \right) u_{\mathbf{p}_1,s} \ . \tag{7.57}$$

Chapter 8

Renormalization

8.1 Loop Expansion of Effective Action

When we calculate the Green's functions following the perturbation theory, we draw various patterns of graphs by contracting external points and perturbatively expanded interaction points. The higher-order graphs, in general, contain some number of closed loops which consist of continuously connected internal lines. Some of the loops produce the divergence in the Green's function. This must be remedied in order to derive the well-defined dynamical contents of the quantum field theory. We found in §6.2 that all of the connected Green's functions are expressed as a sum of "trees" represented in terms of the effective action. Loops are existing only in the effective action. Thus, what we have to do is to establish the systematic method of the remedy for the divergence in the effective action.

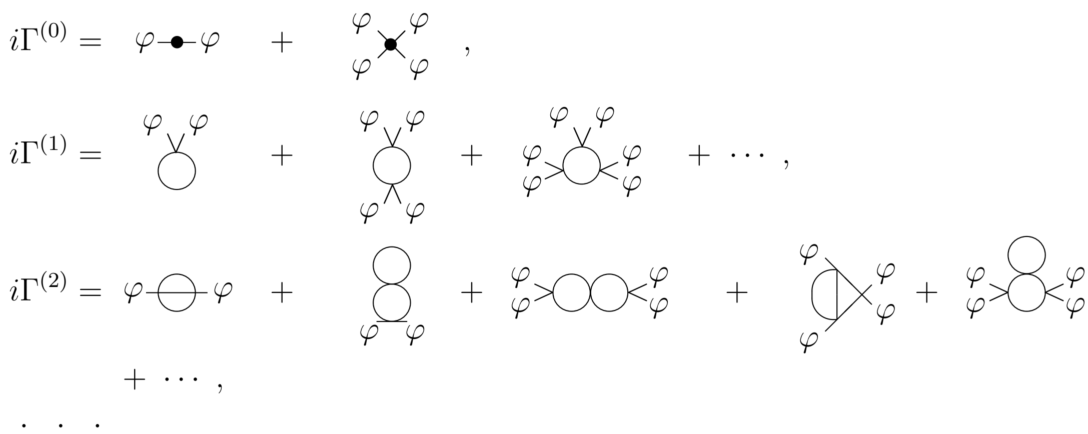
As the simplest example, let us investigate the $\lambda\phi^4$ theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{8.1}$$

We expand the effective action $\Gamma[\varphi]$ of this theory according to the number of loops contained in the graph ;

$$\Gamma[\varphi] = \Gamma^{(0)}[\varphi] + \Gamma^{(1)}[\varphi] + \Gamma^{(2)}[\varphi] + \cdots \ . \tag{8.2}$$

Each of the terms is graphically represented, abbreviating a numerical factor, as



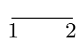
(8.3)

(8.4)

(8.5)

0-Loop $\Gamma^{(0)}$: Let us first calculate the 0-loop $\Gamma^{(0)}[\varphi]$. The 0-loop part of the two-point connected Green's function is just a free field propagator $i\Delta_F(x_1 - x_2)$. From the formula (2.129), it is expressed as

$$G_c^{(0)}(x_1, x_2) =$$



$$= i\Delta_F(x_1 - x_2) = \int d^4x \ i\Delta_F(x_1 - x)i(-\Box - m^2)i\Delta_F(x - x_2) \ . \tag{8.6}$$

Removing of two external 2-point Green's functions $i\Delta_F$ gives the 2-point proper vertex function $i\Gamma^{(0)}(y_1, y_2)$ as

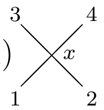
$$i\Gamma^{(0)}(y_1, y_2) = i(-\Box_{(1)} - m^2)\delta^4(y_1 - y_2) \ . \tag{8.7}$$

Therefore, the 2-point part of $\Gamma^{(0)}[\varphi]$ is

$$i\Gamma_2^{(0)}[\varphi] = i\frac{1}{2!}\int d^4x \ \varphi(x)(-\Box - m^2)\varphi(x) \ . \tag{8.8}$$

The four-point connected Green's function is

$$G_c^{(0)}(x_1, \cdots, x_4) = (-i\lambda)$$



$$= -i\lambda \int d^4x \ i\Delta_F(x_1 - x)i\Delta_F(x_2 - x)i\Delta_F(x_3 - x)i\Delta_F(x_4 - x) \ . \tag{8.9}$$

The proper vertex function is

$$i\Gamma^{(0)}(y_1, \cdots, y_4) = -i\lambda\delta^4(y_1 - y_2)\delta^4(y_1 - y_3)\delta^4(y_1 - y_4) \ . \tag{8.10}$$

Therefore, the 4-point part of $\Gamma^{(0)}$ is

$$i\Gamma_4^{(0)}[\varphi] = \frac{1}{4!}(-i\lambda)\int d^4x \ \varphi(x)\varphi(x)\varphi(x)\varphi(x) \ . \tag{8.11}$$

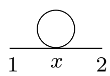
Thus, the 0-loop effective action is

$$\Gamma^{(0)}[\varphi] = \Gamma_2^{(0)}[\varphi] + \Gamma_4^{(0)}[\varphi] = \int d^4x \ \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4\right) \ . \tag{8.12}$$

This is just the classical action integral $I[\varphi]$.

1-Loop $\mathbf{\Gamma^{(1)}}$: At the 1-loop level, we encounter the divergence. The 1-loop part of $G_c(x_1, x_2)$ is

$$G_c^{(1)}(x_1, x_2) = \frac{-i\lambda}{2}$$



$$= \frac{-i\lambda}{2}\int d^4x \ i\Delta_F(x_1 - x)i\Delta_F(x - x)i\Delta_F(x - x_2) \ . \tag{8.13}$$

This graph is called "tadpole" graph. From this expression, we obtain the 2-point part of $\Gamma^{(1)}$ as

$$i\Gamma_2^{(1)}[\varphi] = \frac{1}{2!}\frac{-i\lambda}{2}\int d^4x \ \varphi(x)i\Delta_F(x - x)\varphi(x) \equiv -iD_2^{(1)}\int d^4x \ \frac{1}{2} \ \varphi(x)^2 \ . \tag{8.14}$$

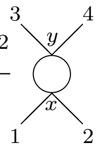
The coefficient $D_2^{(1)}$ is a quadratically divergent constant ;

$$D_2^{(1)} = \frac{\lambda}{2} \ i\Delta_F(x - x) = \frac{\lambda}{2}\lim_{\Lambda\to\infty}\int^\Lambda \frac{d^4q}{(2\pi)^4} \ i \ \frac{1}{q^2 - m^2 + i\epsilon} \propto \Lambda^2 \ , \tag{8.15}$$

where Λ is a ultraviolet cutoff of the $d|q|$ integration. Roughly speaking, $|q|$ is regarded as a inverse-radius of the tadpole, which may be expected from the uncertainty principle $\Delta q\Delta r \simeq 1$. Thus the divergence in (8.15) is lying at the shrinked limit of the tadpole.

The 4-point part is also divergent. The 1-loop connected 4-point function is

$$G_c^{(1)}(x_1, \cdots, x_4) = \frac{(-i\lambda)^2}{2}$$



$$+ \ (1 \leftrightarrow 3) \ + \ (1 \leftrightarrow 4) \ + \ [\text{1-p-r}] \ , \tag{8.16}$$

where [1-p-r] reprsents the irrelevant graphs which contain one-particle reducible internal lines. Thus, we obtain the 4-point part of $\Gamma^{(1)}$ as

$$i\Gamma_4^{(1)}[\varphi] = \frac{1}{4!}\, 3\, \frac{(-i\lambda)^2}{2} \int d^4x d^4y \ \varphi(x)^2 i\Delta_F(x-y) i\Delta_F(x-y)\varphi(y)^2 \ . \tag{8.17}$$

The divergence emerges in the region $x - y \to 0$ where $\Delta_F(x-y)$ becomes singular, as the result (8.15) shows. The isolation of the divergent piece from this expression requires some calculations. The Fourier representation of the coefficient function of the integrand is

$$i\Delta_F(x-y) i\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{q^2 - m^2 + i\epsilon} e^{-i(p-q)\cdot(x-y)} \ . \tag{8.18}$$

Let us pay attention to the fact that the coordinate (x, y) dependence comes only through the combination $(p-q)\cdot(x-y)$. This means that the half of the integrals $d^4p\, d^4q$ produces a coordinate independent function. It is visualized by shifting the integration variable p by $p+q$;

$$i\Delta_F(x-y) i\Delta_F(x-y) = i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} F_4(p^2) \ , \tag{8.19}$$

$$F_4(p^2) \equiv \int \frac{d^4q}{(2\pi)^4}\, i\, \frac{1}{(p+q)^2 - m^2 + i\epsilon} \frac{1}{q^2 - m^2 + i\epsilon} \ . \tag{8.20}$$

In this form of expression, momentum q is called loop momentum. Its flow is confined in the propagators in the loop. In general, l-loop graph admits l independent loop momenta which flow freely within the loops. Now we split the constant term $F_4(0)$ from $F_4(p^2)$ by

$$F_4(p^2) \ = \ F_4(0) + \tilde{F}_4(p^2) \ , \tag{8.21}$$

$$\tilde{F}_4(p^2) \ \equiv \ \int \frac{d^4q}{(2\pi)^4}\, i \left(\frac{1}{(p+q)^2 - m^2 + i\epsilon} - \frac{1}{q^2 - m^2 + i\epsilon} \right) \frac{1}{q^2 - m^2 + i\epsilon} \ , \tag{8.22}$$

and rewrite (8.19) in a form

$$\begin{aligned} \Delta_F(x-y) i\Delta_F(x-y) \ &= \ \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \left(F_4(0) + \tilde{F}_4(p^2) \right) \\ &\equiv \ F_4(0)\delta^4(x-y) + A_4(x-y) \ . \end{aligned} \tag{8.23}$$

The function $\tilde{F}_4(p^2)$ is evidently finite, since the integrand decreases asymptotically faster than $|q|^{-4}$. $F_4(0)$ is a logarithmically divergent constant ;

$$F_4(0) = \lim_{\Lambda \to \infty} \int^{\Lambda} \frac{d^4q}{(2\pi)^4}\, i\, \frac{1}{(q^2 - m^2 + i\epsilon)^2} \propto \log \Lambda \ . \tag{8.24}$$

Again the divergence comes from the shrinked limit of the loop in (8.16). It is reflected as an appearance of $\delta^4(x-y)$ in (8.23). $\Gamma_4^{(1)}$ has a form

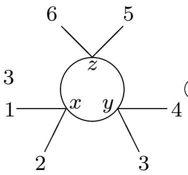
$$i\Gamma_4^{(1)}[\varphi] = \frac{1}{4!}\, 3\, \frac{(-i\lambda)^2}{2} \int d^4x d^4y \ \varphi(x)^2 i A_4(x-y) \varphi(y)^2 - iD_4^{(1)} \int d^4x \ \frac{1}{4!} \ \varphi(x)^4 \ , \tag{8.25}$$

with

$$D_4^{(1)} = \frac{3\lambda^2}{2} F_4(0) \ . \tag{8.26}$$

Let us next examine the 6-point function. The relevant part of the Green's function consists of fifteen pieces corresponding to the different dispositions of x_i's ;

$$G_c^{(1)}(x_1, \cdots, x_6) = (-i\lambda)^3$$



$$\otimes 15 \ [\text{non-equivalent arrangement of } x_i\text{'s}] + [\text{1-p-r}] \ . \tag{8.27}$$

Following the same procedure used in the previous examples, we find for $\Gamma_6^{(1)}$ the expression

$$i\Gamma_6^{(1)}[\varphi] = \frac{1}{6!} 15(-i\lambda)^3 \int d^4x d^4y d^4z \ \varphi(x)^2 \varphi(y)^2 \varphi(z)^2 A_6(x, y, z) \ , \tag{8.28}$$

$$A_6(x, y, z) \equiv i\varDelta_F(x-y) i\varDelta_F(y-z) i\varDelta_F(z-x) \ . \tag{8.29}$$

The Fourier representation of the propagators in $A_6(x, y, z)$ is

$$A_6(x, y, z) = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{r^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y) - iq\cdot(y-z) - ir\cdot(z-x)} \ . \tag{8.30}$$

Let us shift the integration momenta p and q by $p \to q + r, \ q \to q + r$ so that r becomes the loop momentum. Then, $A_6(x, y, z)$ takes a form

$$A_6(x, y, z) = -\int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} e^{-ip\cdot(x-y) - iq\cdot(y-z)} F_6(p, q) \ , \tag{8.31}$$

$$F_6(p, q) \equiv \int \frac{d^4r}{(2\pi)^4} \ i \frac{1}{(p+r)^2 - m^2 + i\epsilon} \frac{1}{(q+r)^2 - m^2 + i\epsilon} \frac{1}{r^2 - m^2 + i\epsilon} \ . \tag{8.32}$$

$F_6(p, q)$ is now a finite function of p and q, since the integrand decreases asymptotically as $|r|^{-6}$. Thus, the $\Gamma_6^{(1)}[\varphi]$ of (8.28), which is the integration over x, y and z, is also finite as was a case in the first term of (8.25).

It is straightforward to extend this result to $\Gamma_{2N}^{(1)}[\varphi]$ with $2N \geq 8$. The one-loop function always has only one loop momentum, say, r and its integrand decreases asymptotically as $|r|^{-2N}$. Therefore, $\Gamma_{2N}^{(1)}[\varphi]$ $(2N \geq 8)$ is always finite.

Prescription for Divergent Integrals : We encountered the divergent integrals of the form

$$I_n \equiv \int \frac{d^4q}{(2\pi)^4} \ i \ \frac{1}{(q^2 - m^2 + i\epsilon)^n} \ ; \qquad n = 1, \ 2 \tag{8.33}$$

in (8.15) and (8.24), where we "intuitively" stated that $I_1 \propto \Lambda^2$ and $I_2 \propto \log \Lambda$ by introducing "not-well-defined" ultraviolet cutoff Λ of the $d|q|$ integration. In order to make them well-defined, we need some prescription for the cutoff Λ. For this purpose, let us first calculate the convergent integrals $I_{n>2}$. From the expression (2.124), we find I_n is expressed as

$$I_n = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left[\int_{-\infty}^{\infty} \frac{dq^0}{2\pi} \ i \ \frac{1}{((q^0 - \omega_q)(q^0 + \omega_q) + i\epsilon)^n} \right] \ ; \qquad \omega_q = \sqrt{\mathbf{q}^2 + m^2} \ . \tag{8.34}$$

Since the integral is convergent and the singularities of the integrand are at $q^0 = \pm(\omega_q - i\epsilon)$, we can safely shift the q^0-integration from $-\infty \to +\infty$ to $-i\infty \to +i\infty$. The integrations over

$-\infty \to -i\infty$ and $+i\infty \to +\infty$ vanish. Therefore the change of the variable $q^0 \Rightarrow iq^4$ gives I_n as a four-dimensional Euclidean space integral

$$I_n = (-1)^{n+1} \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{(q_E^2 + m^2)^n} , \tag{8.35}$$

where

$$d^4 q_E = dq^1 dq^2 dq^3 dq^4 , \qquad q_E^2 = \sum_{a=1}^{4} (q^a)^2 = \mathbf{q}^2 + (q^4)^2 . \tag{8.36}$$

Notice I_n is a real number. This is a consequence of putting i in anticipation in defining I_n in (8.33). Now we define I_1 and I_2 based on (8.35) by introducing a cutoff Λ for $x \equiv |q_E|$:

$$I_n = \lim_{\Lambda \to \infty} (-1)^{n+1} \int d\Omega_4 \int_0^{\Lambda} \frac{x^3 dx}{(2\pi)^4} \frac{1}{(x^2 + m^2)^n} , \tag{8.37}$$

where $d\Omega_4$ is a solid angle of four-dimensional Euclidean space. The total solid angle is given by

$$\int d\Omega_4 = 2\pi^2 . \tag{8.38}$$

Thus, I_1 and I_2 are given by

$$I_1 = \frac{1}{16\pi^2} (\Lambda^2 - m^2 \log(\Lambda^2/m^2)) , \qquad I_2 = -\frac{1}{16\pi^2} (\log(\Lambda^2/m^2) - 1) . \tag{8.39}$$

The convergent integrals $I_{n>2}$ are obtained by differentiating I_2 with respect to m^2 :

$$I_n = \int \frac{d^4 q}{(2\pi)^4} \, i \, \frac{1}{(q^2 - m^2 + i\epsilon)^n} = \frac{1}{(n-1)!} \frac{d^{n-2}}{d(m^2)^{n-2}} I_2 = -\frac{1}{16\pi^2 (n-1)(n-2)} \frac{1}{(-m^2)^{n-2}} . \tag{8.40}$$

Problem (8.1-1) The four-dimensional Euclidean momentum space is parametrized by

$$q^1 = |q_E| \sin\psi \sin\theta \cos\phi , \; q^2 = |q_E| \sin\psi \sin\theta \sin\phi , \; q^3 = |q_E| \sin\psi \cos\theta , \; q^4 = |q_E| \cos\psi , \tag{8.41}$$

where

$$0 \le \psi \le \pi , \quad 0 \le \theta \le \pi , \quad 0 \le \phi \le 2\pi , \quad 0 \le |q_E| < \infty . \tag{8.42}$$

Derive the integration measure

$$d^4 q_E = d\Omega_4 |q_E|^3 d|q_E| ; \qquad d\Omega_4 = \sin^2\psi \, d\psi \, \sin\theta \, d\theta \, d\phi , \tag{8.43}$$

and verify (8.38).

Problem (8.1-2) By replacing m^2 in the expression (8.40) of I_n by $m^2 + p^2$ and shifting the integration momentum q^μ to $q^\mu - p^\mu$, confirm the formula ($n > 2$)

$$\int \frac{d^4 q}{(2\pi)^4} \, i \, \frac{1}{(q^2 - 2p\cdot q - m^2 + i\epsilon)^n} = -\frac{1}{16\pi^2 (n-1)(n-2)} \frac{1}{(-m^2 - p^2)^{n-2}} . \tag{8.44}$$

Further formulas are obtained by differentiating this with respect to p^μ. Verify the formula

$$\int \frac{d^4 q}{(2\pi)^4} \, i \, \frac{q^\mu}{(q^2 - 2p\cdot q - m^2 + i\epsilon)^n} = -\frac{1}{16\pi^2 (n-1)(n-2)} \frac{p^\mu}{(-m^2 - p^2)^{n-2}} . \tag{8.45}$$

Problem (8.1-3) Check the following identity called Feynman's parameter formula ;

$$\frac{1}{AB} = \int_0^1 dx \; \frac{1}{(xA + (1-x)B)^2} \; . \tag{8.46}$$

The derivative of this expression by A generates the second identity

$$\frac{1}{A^2 B} = \int_0^1 dx \; \frac{2x}{(xA + (1-x)B)^3} \; . \tag{8.47}$$

By combining (8.46) and (8.47), confirm the third identity

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \; \frac{2y}{(yxA + y(1-x)B + (1-y)C)^3} \; . \tag{8.48}$$

These identities enable us to carry out the loop momentum integration of $\tilde{F}_4(p^2)$ and $F_6(p,q)$ in (8.22) and (8.32) based on the formulas (8.44) and (8.45). Derive their following expressions :

$$\tilde{F}_4(p^2) = -\frac{1}{16\pi^2} \int_0^1 dx \; \frac{x(1-2x)p^2}{x(1-x)p^2 - m^2} \; , \tag{8.49}$$

$$F_6(p,q) = -\frac{1}{16\pi^2} \int_0^1 dx \int_0^1 dy \; \frac{y}{(yx - y^2x^2)p^2 + \left(y(1-x) - y^2(1-x)^2\right) q^2 - 2y^2 x(1-x) p \cdot q} \; . \tag{8.50}$$

8.2 Renormalization

We are now ready to proceed to the systematic method of remedy for the divergence lying on the effective action $\Gamma[\varphi]$, based on the loop expansion.

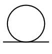
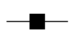
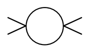
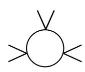
The divergence appearing at the one-loop level was

$$\Gamma^{(1)}_{\rm div}[\varphi] = \int d^4x \; \left(-D_2^{(1)} \frac{1}{2!} \; \varphi(x)^2 - D_4^{(1)} \frac{1}{4!} \; \varphi(x)^4 \right) \; . \tag{8.51}$$

In order to cancel this divergence, we modify the original Lagrangian $\mathcal{L}$ by adding the so-called counter-term $\Delta\mathcal{L}^{(1)}$ and define the new Lagrangian $\mathcal{L}_1$ by

$$\mathcal{L}_1 = \mathcal{L} + \Delta\mathcal{L}^{(1)} \; , \qquad \Delta\mathcal{L}^{(1)} = D_2^{(1)} \frac{1}{2!} \; \phi(x)^2 + D_4^{(1)} \frac{1}{4!} \; \phi(x)^4 \; . \tag{8.52}$$

Using this $\mathcal{L}_1$, we calculate the one-loop $\Gamma^{(1)}[\varphi]$. At this calculation, we treat $\Delta\mathcal{L}^{(1)}$ as a part of interactions $\mathcal{L}_{1\,\rm int}$, though it contains ϕ^2, and assume $\Delta\mathcal{L}^{(1)}$ is a one-loop quantity. The result is

$i\Gamma^{(1)}[\varphi] = \frac{1}{2!}\Big(\frac{1}{2}$  $+$  $\Big) + \frac{1}{4!}\Big(\frac{3}{2}$  $+$  $\Big) + \frac{1}{6!}15$  $+ \cdots \; ,$ (8.53)

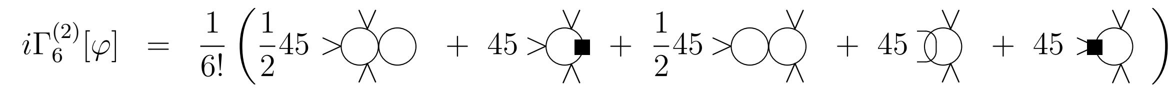
where the black box represents the vertex from $\Delta\mathcal{L}^{(1)}$, and the fields φ's, suppressed in the graphs, should be tacitly attached to the end of lines which we kept for the reminiscence of the removal of the external lines. For example, the third and fourth graphs stand for

$$\frac{1}{4!}\left(\frac{3}{2}(-i\lambda)^2 \int d^4x d^4y \; \varphi(x)^2 \, (i\Delta_F(x-y))^2 \, \varphi(y)^2 + iD_4^{(1)} \int d^4x \; \varphi(x)^4 \right) \; . \tag{8.54}$$

Since $\Delta\mathcal{L}^{(1)}$ is introduced to cancel the divergence of original $\Gamma_2^{(1)} + \Gamma_4^{(1)}$, the sum of the first four graphs of (8.53) is finite. The remaining terms $\Gamma_{2N\geq 6}^{(1)}$ are equivalent to the original $\Gamma_{2N\geq 6}^{(1)}$, which we found to be finite. Therefore, the one-loop $\Gamma^{(1)}[\varphi]$ given by (8.53) is completely finite.

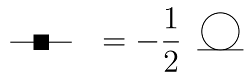
Next step is to calculate the two-loop $\Gamma^{(2)}[\varphi]$ using $\mathcal{L}_1$. At this stage, we will encounter several remarkable facts, which play important roles for the realization of finite $\Gamma[\varphi]$.

The first remarkable fact is that $\Gamma_{2N\geq 6}^{(2)}$ is finite as the one-loop $\Gamma_{2N\geq 6}^{(1)}$ was finite. For example, let us investigate the six point $\Gamma_6^{(2)}[\varphi]$. It is represented as



(8.55)

We realize that the first two graphs exactly cancel out because we introduced the counter-term $\frac{1}{2}D_2^{(1)}\phi^2$ so that it cancels the total of the tadpole ;



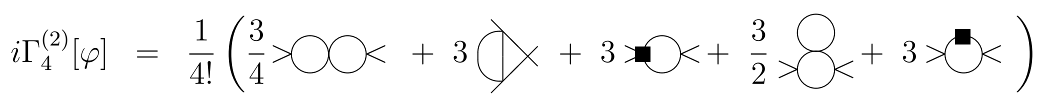
(8.56)

In the third and fourth graphs, the right loops have the same forms as that of one-loop $\Gamma_6^{(1)}$ and $\Gamma_8^{(1)}$, respectively. We already know that these loops do not produce divergence. The divergence emerges from the left loop of each graph. As we found through one-loop calculations, the divergence comes from the shrinked limit of the loop. Therefore, the divergent pieces of the third and fourth graphs have a functional form which precisely coincides with that of the fifth graph. Moreover, as we can confirm from equation (8.53), the counter-term $D_4^{(1)}\frac{1}{4!}\phi^4$ is introduced in the way

(8.57)

Thus, the sum of these three graphs is perfectly finite. For these precise cancelations of the divergences in $\Gamma_6^{(2)}[\varphi]$, the relative weight of each graph is very significant. It is controled by the Wick's expansion theorem. That is, the Wick's theorem has prepared each graph of $\Gamma_6^{(2)}[\varphi]$ so that they sum up to finite. It is straightforward to extend this result to $\Gamma_{2N>6}^{(2)}$. The essence is that, for each graph which contains the divergent loop, we always have a counter graph where the divergent loop is replaced with the black box.

Next, we look at the four point $\Gamma_4^{(2)}[\varphi]$ given by



(8.58)

The last two graphs cancel out due to (8.56). The first graph has two divergent loops. When one of the two loops is shrinked, the resulting form of the graph coincides to that of the third graph. Also, the divergent left loop of the second graph does the same. The Wick's theorem here again has prepared the relative weight of these three graphs so that the cancelation of the divergence of this type proceeds successfully. Let us explicitly write down the first three terms in the parenthesis in detail. From the expression (8.23) and

$$
\begin{aligned}
&\int d^4z\ i\varDelta_F(x-z)^2 i\varDelta_F(z-y)^2 \\
&= 2iF_4(0)\ i\varDelta_F(x-y)^2 - (i^2)F_4(0)^2\delta^4(x-y) + \int d^4z\ iA_4(x-z)\ iA_4(z-y)\ , \quad (8.59)
\end{aligned}
$$

the divergent pieces we are now discussing are collected to be

$$
\left(\frac{3}{4}(-i\lambda)^3 2iF_4(0) + 3(-i\lambda)^3 iF_4(0) + 3iD_4^{(1)}(-i\lambda)\right)\int d^4x d^4y\ \varphi(x)^2\varphi(y)^2\ i\varDelta_F(x-y)^2\ . \quad (8.60)
$$

Each of the terms in the parenthesis comes from each of the first three graphs in (8.58). They sum up to zero owing to (8.26).

We have encountered the two remarkable phenomena, one of whch occured in (8.55) and another in (8.58). $\Gamma^{(2)}[\varphi]$ does not contain the divergent pieces which consist of the non-local products of the fields of the forms

$$[\text{div.}] \times \int d^4x d^4y d^4z\ \varphi(x)^2 \varphi(y)^2 \varphi(z)^2 A_6(x,y,z) \quad \text{and} \quad [\text{div.}] \times \int d^4x d^4y\ \varphi(x)^2 \varphi(y)^2 A_4(x,y)\ . \tag{8.61}$$

This is quite valuable result. If such terms were present in $\Gamma^{(2)}[\varphi]$, we could not remove them by the local counter-term $\Delta\mathcal{L}^{(2)}(\phi(x))$, and consequently it might ruin our prescription based on the loop expansion of $\Gamma[\varphi]$. The Wick's theorem assures us of the absence of such divergences.

Let us proceed to the remaining terms of (8.58). The first graph gives

$$\begin{aligned} &-\frac{3}{4}(i)^2(-i\lambda)^3 F_4(0)^2 \int d^4x\ \varphi(x)^4 \\ &+(-i\lambda)^3 \int d^4x d^4y\ \varphi(x)^2\varphi(y)^2 \int d^4z\ iA_4(x-z)\ iA_4(z-y)\ . \end{aligned} \tag{8.62}$$

The first term has a divergent coefficient proportional to $(\log\Lambda)^2$, but it can be canceled by introducing a two-loop counter-term $\Delta\mathcal{L}^{(2)}(x)$. The second term is finite. The second graph of (8.58) gives

$$3(-i\lambda)^3 \int d^4x d^4y d^4z\ \varphi(x)\varphi(z)\varphi(y)^2 iA_4(x-z) i\Delta_F(x-y) i\Delta_F(z-y)\ . \tag{8.63}$$

The Fourier representation of the coefficient function of the integrand, after shifting the momenta so that q and r become the loop momenta, is

$$\int \frac{d^4p}{(2\pi)^4}\frac{d^4s}{(2\pi)^4} e^{-ip\cdot(x-z)-is\cdot(z-y)} G_4(p,s)\ , \tag{8.64}$$

$$\begin{aligned} G_4(p,s) = &\int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\left(\frac{i}{(p-r+q)^2-m^2+i\epsilon} - \frac{i}{q^2-m^2+i\epsilon}\right) \\ &\times \frac{i}{q^2-m^2+i\epsilon}\frac{i}{r^2-m^2+i\epsilon}\frac{i}{(s-r)^2-m^2+i\epsilon}\ . \end{aligned} \tag{8.65}$$

The function $G_4(p,s)$ is logarithmically divergent. The divergence can be isolated by

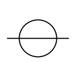
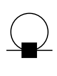
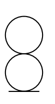
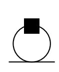
$$G_4(p,s) = G_4(0,0) + \tilde{G}_4(p,s). \tag{8.66}$$

The function $\tilde{G}_4(p,s)$ is finite, and $G_4(0,0) \propto \log\Lambda + (\log\Lambda)^2$. Therefore, the term (8.63) is expressed as

$$3(-i\lambda)^3 G_4(0,0) \int d^4x\ \varphi(x)^4 + 3(-i\lambda)^3 \int d^4x d^4y d^4z\ \varphi(x)\varphi(z)\varphi(y)^2 B_4(x,y,z) \tag{8.67}$$

with the finite function $B_4(x,y,z)$ generated by $\tilde{G}_4(p,s)$ in (8.64). This is the end for $\Gamma_4^{(2)}[\varphi]$.

The remaining task is to investigate the two-point $\Gamma_2^{(2)}[\varphi]$. It is given by

$i\Gamma_2^{(2)}[\varphi] = \frac{1}{2!}\Big(\frac{1}{3!}$  $+ \frac{1}{2}$  $+ \frac{1}{4}$  $+ \frac{1}{2}$  $\Big)$. (8.68)

The sum of the third and fourth graphs vanishes owing to (8.56). The second graph gives divergent ($\propto \Lambda^2 \log \Lambda$) but local operator

$$i\frac{1}{2!}\frac{D_4^{(1)}}{2}i\Delta_F(0)\int d^4x\ \varphi(x)^2\ . \tag{8.69}$$

The first graph is expressed as

$$i\frac{1}{2!}\frac{(-i\lambda)^2}{3!}\int d^4xd^4y\ \varphi(x)\varphi(y)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}G_2(p^2)\ , \tag{8.70}$$

$$G_2(p^2) = \int\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\frac{i}{r^2-m^2+i\epsilon}\frac{1}{(p-q-r)^2-m^2+i\epsilon}\ . \tag{8.71}$$

In order to isolate the divergent pieces, it is useful to expand $G_2(p^2)$ in the Taylor series of p^2 ;

$$G_2(p^2) = G_2(0) + p^2G_2'(0) + \tilde{G}_2(p^2)\ , \quad \tilde{G}_2(p^2) = \frac{1}{2}(p^2)^2G_2''(0) + \cdots\ . \tag{8.72}$$

The first term $G_2(0)$ is a quadratically divergent constant ;

$$G_2(0) = \int\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\frac{i}{r^2-m^2+i\epsilon}\frac{1}{(q+r)^2-m^2+i\epsilon}\propto\Lambda^2\ . \tag{8.73}$$

The coefficient $G_2'(0)$ is determined by expanding the third factor in the integrand of (8.71) ;

$$\begin{aligned}\frac{1}{(p-q-r)^2-m^2+i\epsilon} &= \frac{1}{(q+r)^2-m^2+i\epsilon}\left(1+\frac{p^2-2p\cdot(q+r)}{(q+r)^2-m^2+i\epsilon}\right)^{-1}\\ &= \frac{1}{(q+r)^2-m^2+i\epsilon}-\frac{p^2-2p\cdot(q+r)}{\left((q+r)^2-m^2+i\epsilon\right)^2}+\frac{\left(p^2-2p\cdot(q+r)\right)^2}{\left((q+r)^2-m^2+i\epsilon\right)^3}+\cdots\ .\end{aligned} \tag{8.74}$$

The first contribution comes from p^2 in the second term. $-2p\cdot(q+r)$ does not contribute due to the odd power of the integration variables. The other contribution is $(2p\cdot(q+r))^2$ in the third term. From the symmetry consideration, we recognize that $(q_\mu+r_\mu)(q_\nu+r_\nu)$ is equivalent to $(1/4)g_{\mu\nu}(q+r)^2$ under the integration $d^4q\,d^4r$. Thus, we find

$$\begin{aligned}G_2'(0) = \int\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}&\frac{i}{q^2-m^2+i\epsilon}\frac{i}{r^2-m^2+i\epsilon}\\ &\times\left(-\frac{1}{\left((q+r)^2-m^2+i\epsilon\right)^2}+\frac{(q+r)^2}{\left((q+r)^2-m^2+i\epsilon\right)^3}\right)\ .\end{aligned} \tag{8.75}$$

This is logarithmically divergent, $G_2'(0)\propto\log\Lambda$. The higher terms in the Taylor series $\tilde{G}_2(p^2)$ are convergent. In this way, we obtain $\Gamma_2^{(2)}[\varphi]$ in the form

$$\begin{aligned}i\Gamma_2^{(2)}[\varphi] = i\frac{1}{2!}\frac{(-i\lambda)^2}{3!}G_2'(0)\int d^4x\ \partial_\mu\varphi(x)\partial^\mu\varphi(x) + i\frac{1}{2!}\frac{D_4^{(1)}}{2}i\Delta_F(0)\int d^4x\ \varphi(x)^2\\ +i\frac{1}{2!}\frac{(-i\lambda)^2}{3!}\int d^4xd^4y\ \varphi(x)\varphi(y)B_2(x,y)\end{aligned} \tag{8.76}$$

with $B_2(x,y)$ generated by $\tilde{G}_2(p^2)$. This completes the analysis of $\Gamma^{(2)}[\varphi]$.

All divergences we found in two-loop $\Gamma^{(2)}[\varphi]$ are again the local operators of the form

$$\Gamma^{(2)}_{\rm div}[\varphi] = \int d^4x\ \left(D_K^{(2)}\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - D_2^{(2)}\frac{1}{2}\ \varphi(x)^2 - D_4^{(2)}\frac{1}{4!}\ \varphi(x)^4\right)\ . \tag{8.77}$$

The next step is to redefine the Lagrangian from $\mathcal{L}_1$ to $\mathcal{L}_2$ so that these divergences are canceled,

$$\mathcal{L}_2 = \mathcal{L}_1 + \Delta\mathcal{L}^{(2)} \ , \tag{8.78}$$

$$\Delta\mathcal{L}^{(2)} \equiv -D_K^{(2)}\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + D_2^{(2)}\frac{1}{2}\ \phi(x)^2 + D_4^{(2)}\frac{1}{4!}\ \phi(x)^4 \ , \tag{8.79}$$

assuming, this time, $\Delta\mathcal{L}^{(2)}$ is a two-loop quantity, and continue the same procedure to higher loops. In this way, we expect we will obtain the fully finite effective action $\Gamma[\varphi]$. This is really true. The Wick's expansion theorem always prepares the graphs in each step of the loop expansion so that the non-local divergences are canceled.

The Wick's theorem does not know, in principle, whether each of the graphs is convergent or divergent. What the theorem really works is on the combinatorics. When we find some n-point function is divergent, we introduce n-point counter-term. The combinatorial character of n-point function and n-point counter-term is equivalent. Thus, when we calculate the effective action $\Gamma[\varphi]$ up to l-loops based on the loop expansion, any of the divergences atributed to the lower loop divergences are always canceled by the already introduced counter-terms. The remaining divergences are only at the limit where all of the loops are simultaneously shrinked. Therefore, the resulting operators are always local.

This procedure works for any Lagrangian. The divergences in the l-loop effective action $\Gamma^{(l)}[\varphi]$ are always local

$$\Gamma^{(l)}_{\rm div}[\varphi] = \sum_i D_i^{(l)} \int d^4x \ O_i(\varphi(x)) \tag{8.80}$$

and they are removed by adding counter-terms $\Delta\mathcal{L}^{(l)}$ to the Lagrangian. At this stage, we pose a question. To what extent should we cancel the divergences ? We must decide, in some way, the amount of finite portion, which we preserve for each operators. This is called "renormalization".

There are class of theories, where the number of operators required for the counter-terms is finite. The $\lambda\phi^4$ theory is the typical example. The operators are limited to the three types, $\partial_\mu\phi\partial^\mu\phi$, ϕ^2 and ϕ^4, which are already existing in the original Lagrangian $\mathcal{L}$. Such a theory is called renormalizable. The physical content of the renormalizable theory is expressible in terms of the finite number of parameters, say, m and λ, which charactrize the theory. The nonrenormalizable theory requires infinite number of counter-terms, and it is characterized by infinite number of parameters.

8.3 Renormalization of $\lambda\phi^4$ Theory

Let us proceed further on the $\lambda\phi^4$ theory. The l-th Lagrangian is

$$\mathcal{L}_l = \mathcal{L} + \Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{(2)} + \cdots + \Delta\mathcal{L}^{(l)} \ . \tag{8.81}$$

Taking a limit $l \to \infty$, we define the so-called bare Lagrangian

$$\mathcal{L}_{\rm bare} = \lim_{l\to\infty} \mathcal{L}_l = \mathcal{L} + \sum_{l=1}^{\infty} \Delta\mathcal{L}^{(l)} \ . \tag{8.82}$$

This Lagrangian generates the fully finite effective action $\Gamma[\varphi]$. Substituting the expressions for $\Delta\mathcal{L}^{(l)}$, we find $\mathcal{L}_{\rm bere}$ has a form

$$\mathcal{L}_{\rm bare} = \frac{1}{2}Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_{\rm b}^2 Z\phi^2 - \frac{1}{4!}\lambda_{\rm b}Z^2\phi^4 \ , \tag{8.83}$$

where

$$Z = 1 - \sum_{l=2}^{\infty} D_K^{(l)} \ , \tag{8.84}$$

$$m_{\rm b}^2 Z = m^2 - \sum_{l=1}^{\infty} D_2^{(l)} \ , \tag{8.85}$$

$$\lambda_{\rm b} Z^2 = \lambda - \sum_{l=1}^{\infty} D_4^{(l)} \ . \tag{8.86}$$

Let us now rescale the field ϕ by

$$\phi_{\rm b} = Z^{1/2}\phi \ . \tag{8.87}$$

This is called wave function renormalization. Then, $\mathcal{L}_{\rm bare}$ takes a very simple form

$$\mathcal{L}_{\rm bare} = \frac{1}{2}\partial_\mu \phi_{\rm b} \partial^\mu \phi_{\rm b} - \frac{1}{2} m_{\rm b}^2 \phi_{\rm b}^2 - \frac{1}{4!}\lambda_{\rm b}\phi_{\rm b}^4 \ . \tag{8.88}$$

The parameters $m_{\rm b}$ and $\lambda_{\rm b}$ are called bare mass and bare coupling constant, respectively.

Suppose we calculate the effective action using the Lagrangian (8.88) described by the field operator $\phi_{\rm b}$, introducing the ultraviolet cut-off Λ of the loop momenta. At this calculation, we must not introduce counter-terms because $\mathcal{L}_{\rm bare}$ already contains them. The generating functional of the Green's functions for $\phi_{\rm b}$ is

$$Z_{\rm b}[J_{\rm b}] = e^{iW_{\rm b}[J_{\rm b}]} = \int [d\phi_{\rm b}] e^{i\int d^4x (\mathcal{L}_{\rm bare}(\phi_{\rm b}) + J_{\rm b}\phi_{\rm b})} \ . \tag{8.89}$$

From this, we will derive

$$\varphi_{\rm b}(x) = \frac{\delta W_{\rm b}[J_{\rm b}]}{\delta J_{\rm b}(x)} \ , \qquad \Gamma_{\rm b}[\varphi_{\rm b}] = W_{\rm b}[J_{\rm b}] - \int d^4x \ J_{\rm b}(x)\varphi_{\rm b}(x) \ . \tag{8.90}$$

The resulting $\Gamma_{\rm b}[\varphi_{\rm b}]$ will be expressed in terms of the bare proper vertex functions $\Gamma_{\rm b}(x_1, \cdots, x_N)$ as

$$\Gamma_{\rm b}[\varphi_{\rm b}] = \sum_N \frac{1}{N!}\int d^4x_1 \cdots d^4x_N \ \varphi_{\rm b}(x_1)\cdots\varphi_{\rm b}(x_N)\Gamma_{\rm b}(x_1, \cdots, x_N) \ . \tag{8.91}$$

On the other hand, the same $\mathcal{L}_{\rm bare}$ described by the field operator ϕ in the form (8.82) generates the finite effective action

$$\Gamma[\varphi] = \sum_N \frac{1}{N!}\int d^4x_1 \cdots d^4x_N \ \varphi(x_1)\cdots\varphi(x_N)\Gamma(x_1, \cdots, x_N) \ . \tag{8.92}$$

This is derived from

$$Z[J] = e^{iW[J]} = \int [d\phi] e^{i\int d^4x \left(\mathcal{L}_{\rm bare}(Z^{1/2}\phi) + J\phi\right)} \tag{8.93}$$

through

$$\varphi(x) = \frac{\delta W[J]}{\delta J(x)} \ , \qquad \Gamma[\varphi] = W[J] - \int d^4x \ J(x)\varphi(x) \ . \tag{8.94}$$

Comparing (8.89) and (8.93), we find

$$Z_{\rm b}[J_{\rm b}] = Z[Z^{1/2}J_{\rm b}] \tag{8.95}$$

up to irrelevant overall normalization factor. Therefore, if we set $J_{\rm b} = Z^{-1/2}J$, we must have

$$\varphi_{\rm b} = Z^{1/2}\varphi \ , \qquad \Gamma_{\rm b}[\varphi_{\rm b}] = \Gamma_{\rm b}\left[Z^{1/2}\varphi\right] = \Gamma\left[\varphi\right] \ . \tag{8.96}$$

This result declares the far-reaching conclusion

$$Z^{N/2}\Gamma_{\rm b}(x_1, \cdots, x_N) = \Gamma(x_1, \cdots, x_N) \ . \tag{8.97}$$

The bare proper vertex function $\Gamma_{\rm b}(x_1, \cdots, x_N)$ is a function of $m_{\rm b}$, $\lambda_{\rm b}$ and cut-off Λ, as well as of the space-time points x_i. The equation (8.97) states that the most of the Λ-dependences are absorbed to $m_{\rm b}$ and $\lambda_{\rm b}$, which, as a result, turn out to be a finite m and λ. The remaining Λ-dependences are removed by the multiplicative wave function renormalization factor $Z^{N/2}$.

As we found in §7.3, the physical mass of the particle $m_{\rm phys}$ is a position of the pole of the Fourier integrand of the two-point connected Green's function. In a word of the proper vertex function, it is the zero point of the integrand of the Fourier representation

$$\Gamma_{\rm b}(x, y) = \int \frac{d^4p}{(2\pi)^4}\Gamma_{\rm b}(p^2)e^{-ip\cdot(x-y)} \ . \tag{8.98}$$

(Here, we are tacitly assuming, for simplicity, that $\langle 0|\hat{\phi}_{\rm bare}|0\rangle = 0$). Therefore, the condition

$$\Gamma_{\rm b}(m_{\rm phys}^2) = 0 \tag{8.99}$$

determines $m_{\rm b}$ as a function of $m_{\rm phys}$, $\lambda_{\rm b}$ and Λ. The wave function renormalization factor Z is also determined by expanding $\Gamma_{\rm b}(p^2)$ around $p^2 = m_{\rm phys}^2$;

$$\Gamma_{\rm b}(p^2) = Z^{-1}(p^2 - m_{\rm phys}^2) + O\left((p^2 - m_{\rm phys}^2)^2\right) \ . \tag{8.100}$$

If we represent $Z^{N/2}\Gamma_{\rm b}(x_1, \cdots, x_N)$ as a function of $m_{\rm phys}$, $\lambda_{\rm b}$ and Λ, using the obtained expressions of $m_{\rm b}$ and Z, the surviving Λ-dependences are concentrated around $\lambda_{\rm b}$ so that they can be absorbed to $\lambda_{\rm b}$ and disappear from $Z^{N/2}\Gamma_{\rm b}(x_1, \cdots, x_N)$. The bare coupling constant $\lambda_{\rm b}$ is fixed by imposing some condition on the integrand of the Fourier representation of the four-point proper vertex function

$$\Gamma_{\rm b}(x_1, \cdots, x_4) = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\Gamma_{\rm b}(p, q, r)e^{-ip\cdot(x_1-x_2)-iq\cdot(x_1-x_3)-ir\cdot(x_1-x_4)} \ . \tag{8.101}$$

Notice that the 0-loop function (8.10) is expressed as

$$\Gamma^{(0)}(x_1, \cdots, x_4) = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(-\lambda)e^{-ip\cdot(x_1-x_2)-iq\cdot(x_1-x_3)-ir\cdot(x_1-x_4)} \ , \tag{8.102}$$

and divergences due to the loop corrections are also local and have a same functional form. Therefore, the condition we should impose is on the constant part of $\Gamma_{\rm b}(p, q, r)$, which may depend on its definition. The simplest example of the condition will be

$$Z^2\Gamma_{\rm b}(0, 0, 0) = -\lambda_{\rm phys} \ . \tag{8.103}$$

As a result, all of $m_{\rm b}$, $\lambda_{\rm b}$ and Z are expressed as a function of $m_{\rm phys}$, $\lambda_{\rm phys}$ and cut-off Λ, and Λ dependence disappears from

$$Z^{N/2}\Gamma_{\rm b}(x_1, \cdots, x_N) = \Gamma(x_1, \cdots, x_N) \ . \tag{8.104}$$

The function $\Gamma(x_1, \cdots, x_N)$ is called renormalized vertex function. It contains only physical parameters $m_{\rm phys}$, $\lambda_{\rm phys}$, and these parameters characterize the theory. It gives a building brock for all of the Green's functions.

Chapter 9

Symmetry and Effective Potential

9.1 Symmetry of Effective Action

Symmetry : Let $\mathcal{L}(\phi, \phi^\dagger)$ be the Lagrangian of the interacting complex scalar fields ϕ and $\phi^\dagger$:

$$\mathcal{L}(\phi, \phi^\dagger) = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2 \ . \tag{9.1}$$

This Lagrangian is invariant under the following phase transformation of the fields with the space-time independent transformation parameter θ ;

$$\phi \longrightarrow \phi' = e^{i\theta} \phi \ , \quad \phi^\dagger \longrightarrow \phi'^\dagger = e^{-i\theta} \phi^\dagger \ , \tag{9.2}$$

$$\mathcal{L} \longrightarrow \mathcal{L}(\phi', \phi'^\dagger) = \mathcal{L}(\phi, \phi^\dagger) \ . \tag{9.3}$$

If we represent the complex field ϕ in terms of the two real fields ϕ^1 and ϕ^2 as $\phi = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)$, the transformation (9.2) is

$$\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \phi'^1 \\ \phi'^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \ . \tag{9.4}$$

Let us generalize this example to the system with N components of real scalar fields ϕ^i ($i = 1, \cdots, N$). When the system contains complex scalar fields, we express them in terms of the couple of real scalar fields. Now, we introduce the $N \times N$ transformation matrix U by

$$\phi^i \to \phi'^i = \sum_{j=1}^{N} U^{ij} \phi^j \quad : \qquad \phi \to \phi' = U\phi \ , \tag{9.5}$$

and assume that the classical action integral $I[\phi]$ is invariant under this transformation ;

$$I[\phi] \to I[\phi'] = I[U\phi] = I[\phi] \ . \tag{9.6}$$

The matrix U must be orthogonal ($U^T U = 1$) so that the transformation (9.5) keeps the so-called kinetic terms $\sum_{i=1}^{N} \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i$ of the scalar Lagrangian invariant. Any orthogonal matrix U can be expressed as an exponential of an antisymmetric matrix X ;

$$U = e^X \quad : \qquad X^T = -X \ . \tag{9.7}$$

We further assume that $I[\phi]$ is invariant under the continuous transformation characterized by the matrix

$$U = e^{tX} \tag{9.8}$$

with an arbitrary real t.

Let $\mathcal{G}$ be the set of antisymmetric matrix X which realizes $I[e^X\phi] = I[\phi]$;

$$\mathcal{G} = \{X : I[e^X\phi] = I[\phi]\} . \tag{9.9}$$

It is straightforward to show

$$\text{If } X \in \mathcal{G}, \qquad \text{then } \; tX \in \mathcal{G} , \tag{9.10}$$

$$\text{If } X_1, X_2 \in \mathcal{G}, \quad \text{then } \; X_1 + X_2 \in \mathcal{G} \;,\; [X_1, X_2] \in \mathcal{G} . \tag{9.11}$$

The set $\mathcal{G}$ is called the Lie algebra of the transformation group G defined by the set

$$G = \{U : I[U\phi] = I[\phi]\} . \tag{9.12}$$

Let T_a $(a = 1, \cdots, d_G)$ be the basis matrix of $\mathcal{G}$ satisfying $\mathrm{tr}(T_aT_b) \propto \delta_{ab}$, which is called generators of G. Since, $[T_a, T_b] \in \mathcal{G}$, $[T_a, T_b]$ can be expanded by T_c as

$$[T_a, T_b] = \sum_{c=1}^{d_G} f^{abc} T_c \tag{9.13}$$

with the real constants f^{abc} which characterize the structure of the group G.

The infinitesimal transformation of (9.5) with the infinitesimal real parameters ϵ^a is

$$\phi^i \longrightarrow \phi'^i = \phi^i + \delta\phi^i \;, \qquad \delta\phi^i = \sum_{a=1}^{d_G} \sum_{j=1}^{N} \epsilon^a T_a^{ij} \phi^j . \tag{9.14}$$

The change of $I[\phi]$ under this transformation is formally expressed as

$$I[\phi'] \;=\; I[\phi] + \int d^4x \sum_i \frac{\delta I[\phi]}{\delta\phi^i(x)} \delta\phi^i(x) = I[\phi] + \sum_{a=1}^{d_G} \epsilon^a \int d^4x \sum_{i,j=1}^{N} \frac{\delta I[\phi]}{\delta\phi^i(x)} T_a^{ij} \phi^j(x) . \tag{9.15}$$

The invarinace of $I[\phi]$ thus gives the identity relation

$$\int d^4x \sum_{i,j} \frac{\delta I[\phi]}{\delta\phi^i(x)} T_a^{ij} \phi^j(x) = 0 . \tag{9.16}$$

The invariance of the system is reflected in the generating functional of the Green's functions

$$Z[J] = \int [d\phi] e^{iI[\phi] + i\int d^4x \sum_i J^i(x)\phi^i(x)} . \tag{9.17}$$

Replace the integration variables ϕ^i in this exression by $\phi'^i = \phi^i + \epsilon^a T_a^{ij} \phi^j$. Here and hereafter, for the sake of simplification of the expression, we omit the summation symbol Σ for the repeated indices. Since $[d\phi'] = [\det U d\phi] = [d\phi]$, (9.17) is expressed as

$$\begin{aligned} Z[J] &= \int [d\phi'] e^{iI[\phi'] + i\int d^4x J^i \phi'^i} \\ &= \int [d\phi] e^{iI[\phi] + i\int d^4x J^i(\phi^i + \epsilon^a T_a^{ij} \phi^j)} \\ &= \int [d\phi] \left(1 + i \int d^4x \; J^i(x) \epsilon^a T_a^{ij} \phi^j(x)\right) e^{iI[\phi] + i\int d^4x J^i \phi^i} \\ &= Z[J] + i\epsilon^a \int d^4x \; J^i(x) T_a^{ij} \int [d\phi] \phi^j(x) e^{iI[\phi] + i\int d^4x J^i \phi^i} . \end{aligned} \tag{9.18}$$

Therefore, we obtain

$$0 = \int d^4x \; J^i(x) T_a^{ij} \frac{\delta}{i\delta J^j(x)} Z[J] \; . \tag{9.19}$$

This is the representation of the symmetry of the system expressed in terms of the generating functional $Z[J]$. This is translated into that of the effective action $\Gamma[\varphi]$. From the relations

$$\frac{\delta Z[J]}{i\delta J^i(x)} = \frac{\delta W[J]}{\delta J^i(x)} Z[J] = \varphi^i(x) Z[J] \; , \qquad J^i(x) = -\frac{\delta \Gamma[\varphi]}{\delta \varphi^i(x)} \; , \tag{9.20}$$

we find

$$\int d^4x \; \frac{\delta \Gamma[\varphi]}{\delta \varphi^i(x)} T_a^{ij} \varphi^j(x) = 0 \; . \tag{9.21}$$

This result asserts that the effective action $\Gamma[\varphi]$ is invariant under the transformation which has the same form as that of the original one given by (9.14) ;

$$\varphi^i \longrightarrow \varphi'^i = \varphi^i + \delta\varphi^i \; , \qquad \delta\varphi^i = \epsilon^a T_a^{ij} \varphi^j \; . \tag{9.22}$$

That is, the effective action inherits the symmetry of the classical action $I[\phi]$ without any modification.

Addendum : It is straightforward to extend the consideration to the case where the system contains Dirac fields ψ^m as well as scalar fields ϕ^i. Suppose the classical action is invariant under the transformation

$$\delta\psi^m = \epsilon^a T_a^{mn} \psi^n \; , \qquad \delta\phi^i = \epsilon^a T_a^{ij} \phi^j \; . \tag{9.23}$$

The generators T_a^{mn} for Dirac fields must also satisfy the commutation relations (9.13) so that the transformation group G is identical both for ψ and ϕ, but are now anti-hermitian becouse ψ's are complex fields. The effective action $\Gamma[\psi, \varphi]$ of this system is again invariant under the same transformation as (9.23). Strictly speaking, the general statement given below (9.22) has an exception. The exception occurs when the system contains so-called gauge fields A^μ as well as ψ. In such a case, we happen to have a transformation which preserves a classical action invariant but does not realize the equality $[d\psi'] = [d\psi]$, which, as a result, ruins the above derivation. It is generally called quantum anomaly. Typical example is a so-called chiral anomaly.

9.2 Effective Potential

The effective action $\Gamma[\varphi]$ is a functional of $\varphi(x)$. If we set $\varphi(x) = \varphi$, a constant all over the space-time, $\Gamma[\varphi]$ is expressed in a form

$$\Gamma[\varphi] = -\int d^4x \; V_{\text{eff}}(\varphi) = -\lim_{T,V\to\infty} T \cdot V \cdot V_{\text{eff}}(\varphi) \; . \tag{9.24}$$

$V_{\text{eff}}(\varphi)$ can be interpreted as a "potential energy" per unit volume. It is called effective potential. As we found in §6.2, the vaccum expectation values $\langle 0|\hat{\phi}^i(x)|0\rangle$ are determined as a solution of $\frac{\delta\Gamma[\varphi]}{\delta\varphi^i(x)} = 0$. Owing to the translational invariance of the vacuum, $\langle 0|\hat{\phi}^i(x)|0\rangle$ should be independent of the space-time point x. Therefore, this condition is translated to

$$\frac{\partial V_{\text{eff}}(\varphi)}{\partial \varphi^i} = 0 \; . \tag{9.25}$$

That is, the minimum point of the effective potential $V_{\text{eff}}(\varphi)$ determines the vacuum expectation value

$$\varphi_0^i = \langle 0|\hat{\phi}^i(x)|0\rangle \; . \tag{9.26}$$

As we found in §8.1, the contribution of 0-loop graphs to $\Gamma[\varphi]$ is the classical action integral. Therefore, if we neglect the contribution from higher loop graphs, $V_{\text{eff}}(\varphi)$ reduces to the classical potential $V(\varphi)$ appearing in the original Lagrangian ;

$$V_{\text{eff}}(\varphi) \underset{0-\text{loop}}{\Longrightarrow} V(\varphi) \ . \tag{9.27}$$

Therefore, at the 0-loop level of the perturbation theory, the vacuum expectation values are determined by minimizing the classical potential $V(\varphi)$. The higher loop effects, in general, shift the value of φ_0^i. The shift can be systematically calculated based on the loop expansion.

9.3 Spontaneous Symmetry Breakdown

Let us investigate the dynamical system which has a continuous symmetry discussed in the previous section. The effective potential also has a symmetry (omitting the index i of φ^i)

$$V_{\text{eff}}(\varphi) \longrightarrow V_{\text{eff}}(\varphi') = V_{\text{eff}}(\varphi) \qquad : \ \varphi' = \varphi + \epsilon^a T_a \varphi \ . \tag{9.28}$$

When the minimum of $V_{\text{eff}}(\varphi)$ is located at a point $\varphi = 0$, the vacuum expectation value of $\hat{\phi}$ is uniquely determined as

$$\langle 0|\hat{\phi}|0\rangle = \varphi_0 = 0 \ . \tag{9.29}$$

Even if we transform this φ_0 according to the transformation (9.28), $\varphi_0 = 0$ is unchanged. That is, the vacuum is invariant under the transformation of the symmetry with which the system is invested. This situation is called symmetric phase or unbroken phase.

Let us next consider the alternative case where $\varphi = 0$ is not a minimum point of $V_{\text{eff}}(\varphi)$. Suppose $\varphi = \varphi_0 \neq 0$ gives the minimum value of $V_{\text{eff}}(\varphi)$. The symmetry (9.28) immediately states that the infinitesimally transformed $\varphi_0' = \varphi_0 + \epsilon^a T_a \varphi_0$ equally gives the minimum value of $V_{\text{eff}}(\varphi)$. Evidently, $\varphi_0'' = \varphi_0' + \epsilon^a T_a \varphi_0'$ also does so. Iterating this transformation N times and taking a limit $N \to \infty$ with keeping $\theta^a \equiv N\epsilon^a$ finite, we find that all of

$$\varphi_0(\theta) \equiv \lim_{N\to\infty} \left(1 + \epsilon^a T_a\right)^N \varphi_0 = \lim_{N\to\infty} \left(1 + \frac{\theta^a}{N} T_a\right)^N \varphi_0 = e^{\theta^a T_a} \varphi_0 \tag{9.30}$$

give the minimum value of $V_{\text{eff}}(\varphi)$ for any values of θ^a. Therefore the vacuum, which is characterized by the minimum of $V_{\text{eff}}(\varphi)$, is not unique. There are infinitely degenerate vacuua. Among these infinite vacuua, one vacuum is chosen at random and it plays a role of the vacuum of the dynamical system. As a result, the vacuum expectation value $\varphi_0 = \langle 0|\hat{\phi}|0\rangle$ moves under the symmetry transformation of the dynamical system. In other words, the vacuum does not maintain the symmetry of the dynamics. This phenomenon is called spontaneous breakdown of the symmetry. The vacuum is called to be realized in the broken phase.

9.4 Nambu-Goldstone Theorem

When the dynamical system has a continuous symmetry, its effective potential $V_{\text{eff}}(\varphi)$ satisfies the identity relation

$$\frac{\partial V_{\text{eff}}(\varphi)}{\partial \varphi^i} T_a^{ij} \varphi^j = 0 \ . \tag{9.31}$$

The derivative of this expression with respect to φ^k produces

$$\frac{\partial^2 V_{\text{eff}}(\varphi)}{\partial \varphi^k \partial \varphi^i} T_a^{ij} \varphi^j + \frac{\partial V_{\text{eff}}(\varphi)}{\partial \varphi^i} T_a^{ik} = 0 \ . \tag{9.32}$$

At the vacuum $\varphi^i = \varphi_0^i$, we know the equation

$$\left.\frac{\partial V_{\text{eff}}(\varphi)}{\partial \varphi^i}\right|_{\varphi_0} = 0 \tag{9.33}$$

is satisfied. Therefore, we obtain

$$\left.\frac{\partial^2 V_{\text{eff}}(\varphi)}{\partial \varphi^k \partial \varphi^i}\right|_{\varphi_0} T_a^{ij} \varphi_0^j = 0 \ . \tag{9.34}$$

Let us define the fields $\eta^i(x)$ by

$$\varphi^i(x) = \varphi_0^i + \eta^i(x) \ , \tag{9.35}$$

which describe the freedom of φ^i around the vacuum expectation values φ_0^i, and expand $V_{\text{eff}}(\varphi)$ around φ_0^i ;

$$V_{\text{eff}}(\varphi) = V_{\text{eff}}(\varphi_0) + \left.\frac{\partial V_{\text{eff}}}{\partial \varphi^i}\right|_{\varphi_0} \eta^i + \frac{1}{2} \left.\frac{\partial^2 V_{\text{eff}}(\varphi)}{\partial \varphi^i \partial \varphi^j}\right|_{\varphi_0} \eta^i \eta^j + \cdots \ . \tag{9.36}$$

The second term simply vanishes owing to (9.33). The third term, which is quadratic in η^i, represents the mass term of η^i. Therefore, the coefficient

$$\left.\frac{\partial^2 V_{\text{eff}}(\varphi)}{\partial \varphi^i \partial \varphi^j}\right|_{\varphi_0} \equiv m_{ij}^2 \tag{9.37}$$

is a mass-square matrix of η^i.

Now we go back to the equation (9.34). It shows the remarkable fact. When $\varphi_0 = 0$ and the symmetry is preserved in the vacuum, it is a trivial equation, $0 = 0$. But once the symmetry is spontaneously broken, it states that the mass matrix m_{ij}^2 of η^i contains the zero-eigenvalues with eigenvectors given by

$$\mathcal{T}_a^i = T_a^{ij} \varphi_0^j \ . \tag{9.38}$$

That is, the spontaneous breakdown of the continuous symmetry leads to the appearance of the massless scalar particles. They are called Nambu-Goldstone bosons. The number of Nambu-Goldstone bosons is the number of linearly independent eigenvectors $\mathcal{T}_a^i$. As we find below, this number coincides to the rank of the real symmetric matrix

$$\Pi_{ab} \equiv \mathcal{T}_a^i \mathcal{T}_b^i \ . \tag{9.39}$$

Suppose Π_{ab} has M eigenvectors U_A^a $(A = 1, \cdots, M)$ with zero eigenvalue ;

$$\Pi_{ab} U_A^b = 0 \qquad A = 1, \cdots, M \ . \tag{9.40}$$

This implies

$$U_A^a \mathcal{T}_a^i = U_A^a T_a^{ij} \varphi_0^j = 0 \tag{9.41}$$

because it leads to

$$0 = U_A^a \Pi_{ab} U_A^b = U_A^a \mathcal{T}_a^i \mathcal{T}_b^i U_A^b = \sum_i (U_A^a \mathcal{T}_a^i)^2 \ , \tag{9.42}$$

where A is not summed. (9.41) means that the transformation by the new generators $T_A = U_A^a T_a$ preserves φ_0 invariant. That is, the symmetry corresponding to T_A is not broken.

The remaining eigenvectors V_K^a of Π_{ab} have nonzero eigenvalues π^K $(K = 1, \cdots, d_G - M)$

$$\Pi_{ab} V_K^b = \pi^K V_K^a \qquad K = 1, \cdots, d_G - M \ . \tag{9.43}$$

This time, we realize

$$V_K^a \mathcal{T}_a^i = V_K^a T_a^{ij} \varphi_0^j \neq 0 \ , \tag{9.44}$$

and consequently the symmetry corresponding to $T_K = V_K^a T_a$ is broken. (9.44) gives the $d_G - M$ eigenvectors of the mass matrix m_{ij}^2 with vanishing eigenvalue. From this result, we find that the massless Nambu-Goldstone bosons appear in one-to-one correspondence to the broken generators of the symmetry. This is called Nambu-Goldstone theorem.

9.5 Chapter-Project/T

Suppose the dynamical system consists of two flavor Dirac fields ψ_i $(i = 1, 2)$ and three real scalar fields ϕ_a $(a = 1, 2, 3)$. We express them collectively by

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \bar{\psi} = \left(\bar{\psi}_1 \ \bar{\psi}_2\right), \qquad \phi = \sum_{a=1}^{3} \phi_a \tau^a, \tag{9.45}$$

where τ^a $(a = 1, 2, 3)$ are Pauli matrices, and therefore ϕ is a 2×2 hermitian traceless matrix. The Lagrangian of the system we discuss is

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - g \bar{\psi} \phi \psi + \frac{1}{4} \mathrm{tr}\left(\partial_\mu \phi \partial^\mu \phi\right) - \frac{\mu^2}{4} \mathrm{tr}\left(\phi\phi\right) - \frac{\lambda}{16} \mathrm{tr}\left(\phi\phi\phi\phi\right). \tag{9.46}$$

The coupling constants g, $\lambda (> 0)$, and scalar mass-square μ^2 are real parameters. We assume the Dirac field ψ has no mass term.

Let us examine the symmetry of the model. Let U be the constant 2×2 unitary matrix $(UU^\dagger = U^\dagger U = 1)$ and difine the flavor transformation

$$\psi \to U\psi, \qquad \bar{\psi} \to \bar{\psi} U^\dagger, \qquad \phi \to U \phi U^\dagger. \tag{9.47}$$

Evidently, (9.46) is invariant under this transformation. This symmetry is called $U(2)$ symmetry. The transformation matrix U is parametrized by four real parameters θ_0, θ_a $(a = 1, 2, 3)$;

$$U(\theta) = e^{i\theta} \qquad : \ \theta = \theta_0 + \sum_{a=1}^{3} \theta_a \tau^a. \tag{9.48}$$

Owing to the massless nature of ψ, the Lagrangian (9.46) is invested with the additional symmetry. The last three terms in (9.46) are invariant under

$$\phi \to -\phi. \tag{9.49}$$

It is possible to define the transformation of ψ so that the first and second terms are also invariant. It needs some preparation. Define the fifth Dirac matrix γ^5 by

$$\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3. \tag{9.50}$$

It is straightforward to show that this matrix is hermitian and anticommutes with all γ^μ $(\mu = 0, 1, 2, 3)$;

$$\gamma^{5\dagger} = \gamma^5, \qquad \{\gamma^5, \gamma^\mu\} = 0 \ \ (\mu = 0, 1, 2, 3). \tag{9.51}$$

It also satisfies

$$(\gamma^5)^2 = 1. \tag{9.52}$$

The transformation for ψ is

$$\psi \to \gamma^5 \psi. \tag{9.53}$$

Then $\bar{\psi}$ transforms as

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 \to (\gamma^5 \psi)^\dagger \gamma^0 = \psi^\dagger \gamma^5 \gamma^0 = -\bar{\psi}\gamma^5, \tag{9.54}$$

and we have

$$\bar{\psi}\psi \to -\bar{\psi}\gamma^5\gamma^5\psi = -\bar{\psi}\psi, \qquad \bar{\psi}\gamma^\mu\psi \to -\bar{\psi}\gamma^5\gamma^\mu\gamma^5\psi = \bar{\psi}\gamma^\mu\psi. \tag{9.55}$$

Consequently the transformation defined by (9.49) and (9.53) preserves (9.46) invariant. We name this transformation R. Since the successive two times action of R leads to the identity operation ($R^2 = 1$), this symmetry is called Z_2.

Thus we realize the dynamical system described by (9.46) has a symmetry $U(2) \times Z_2$.

Let us examine a tree-level analysis of the model. The effective potential at the tree level is

$$V(\phi) = \frac{\mu^2}{4} \mathrm{tr}\,(\phi\phi) + \frac{\lambda}{16} \mathrm{tr}\,(\phi\phi\phi\phi)\,. \tag{9.56}$$

Since ϕ is the 2×2 hermitian traceless matrix, its vacuum expectation value is diagonalized by the unitary transformation in the form

$$\langle 0|\phi|0\rangle = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \tag{9.57}$$

with real $v \geq 0$. v is determined to minimize

$$V(v) = \frac{\mu^2}{2} v^2 + \frac{\lambda}{8} v^4. \tag{9.58}$$

λ must be positive so that the model has a stable vacuum. When $\mu^2 > 0$, $v = 0$ gives the minimum of $V(v)$. This vacuum is invariant under the transformation of $U(2) \times Z_2$.

When $\mu^2 < 0$, the minimum of $V(v)$ is realized at

$$v = \sqrt{-\frac{2\mu^2}{\lambda}}, \tag{9.59}$$

and both the $U(2)$ and Z_2 symmetries are spontaneously broken. The unbroken symmetry transformation in $U(2)$ which preserves the vacuum expectation value (9.57) unchanged under (9.47) is

$$U_U = e^{i\theta_U} \qquad : \ \theta_U = \theta_0 + \theta_3 \tau^3. \tag{9.60}$$

The broken generators are τ^1 and τ^2, and we have two massless Nambu-Goldstone bosons.

Let us express the dynamical degrees of freedom of ϕ around the vacuum expectation value by

$$\phi(x) = \begin{pmatrix} v + \eta(x) & \sqrt{2}\pi^+(x) \\ \sqrt{2}\pi^-(x) & -v - \eta(x) \end{pmatrix}, \tag{9.61}$$

where η is a real scalar and $\pi^\pm = (\pi^\mp)^\dagger$ are complex scalars. The potential $V(\phi)$ takes a form

$$V(\phi) = -\frac{\lambda v^4}{8} + \frac{\lambda v^2}{2}\eta^2 + \lambda v \left(\pi^+\pi^- + \frac{\eta^2}{2}\right)\eta + \lambda \left(\pi^+\pi^- + \frac{\eta^2}{2}\right)^2. \tag{9.62}$$

We find the massless Nambu-Goldstone bosons are π^+ and π^-. The mass of a real scalar particle η is

$$m_\eta = \sqrt{\lambda}\, v. \tag{9.63}$$

The unbroken continuous symmetry is exhausted by (9.60), that is $U(1)\times U(1)$. Let us examine the operation of a broken element

$$U_B = e^{i\theta_B} \qquad : \ \theta_B = \theta_2\tau^2. \tag{9.64}$$

The vacuum expectation value (9.57), which is proportional to τ^3, transforms as

$$\begin{aligned} U_B\tau^3 U_B^\dagger &= \tau^3 + i\theta_2[\tau^2,\tau^3] + \frac{(i\theta_2)^2}{2!}[\tau^2,[\tau^2,\tau^3]] + \frac{(i\theta_2)^3}{3!}[\tau^2,[\tau^2,[\tau^2,\tau^3]]] + \cdots \\ &= \tau^3\cos 2\theta_2 - \tau^1 \sin 2\theta_2, \end{aligned} \tag{9.65}$$

Therefore the transformation U_B with $\theta_2 = \frac{\pi}{2}$ reverses the sign of (9.57). It is convenient to multiply a harmless overall phase factor with $\theta_0 = \pi/2$. The transformation matrix is given by

$$U_B(\theta_0 = \theta_2 = \pi/2) = i\left(\cos\theta_2 + i\tau^2\sin\theta_2\right) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \equiv J \tag{9.66}$$

Remember that the transformation R given by (9.49) also changes the sign of (9.57). Thus the combined transformation of R and J, which we call S, keeps the vacuum invariant.

Under the S transformation, ψ and ϕ transform as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \to \quad J\begin{pmatrix} \gamma^5\psi_1 \\ \gamma^5\psi_2 \end{pmatrix} = \begin{pmatrix} i\gamma^5\psi_2 \\ -i\gamma^5\psi_1 \end{pmatrix}, \tag{9.67}$$

$$\phi = \begin{pmatrix} v+\eta & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -v-\eta \end{pmatrix} \quad \to \quad -J\phi J^\dagger = \begin{pmatrix} v+\eta & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -v-\eta \end{pmatrix}. \tag{9.68}$$

It is convenient to redefine the components of ψ by

$$\psi_u \equiv \psi_1, \qquad \psi_d \equiv i\gamma^5\psi_2. \tag{9.69}$$

The S transformation is

$$\psi_u \leftrightarrow \psi_d, \qquad \pi^+ \leftrightarrow \pi^- \tag{9.70}$$

with η unchanged. Since $J^2 = R^2 = 1$, the symmetry of S transformation is again Z_2. The symmetry breaking pattern of the model is

$$U(2)\times Z_2 \to U(1)\times U(1)\times Z_2'. \tag{9.71}$$

The fermionic part of the Lagrangian is

$$\mathcal{L}_\psi = \sum_{l=u,d}\bar\psi_l(i\gamma^\mu\partial_\mu - m)\psi_l - g\left((\bar\psi_u\psi_u + \bar\psi_d\psi_d)\eta + \sqrt{2}(\bar\psi_u i\gamma^5\psi_d\pi^+ + \bar\psi_d i\gamma^5\psi_u\pi^-)\right), \tag{9.72}$$

with $m = gv$. Notice that the originally massless fermions ψ's acquire the common nonvanishing mass m through the spontaneous symmetry breaking of R. The equality of their masses ($m_u = m_d$) is guaranteed by the unbroken symmetry S.

Appendix A

Perturbation Theory Based on the Interaction Picture

A.1 Interaction Picture

We give in this appendix a brief survey of the perturbation theory based on the interaction picture.

Suppose a state vector $|\psi(t)\rangle$ undergoes a time development according to the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{A.1}$$

with the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V} \ . \tag{A.2}$$

The state vector in the interaction picture $|\psi(t)\rangle_{\rm I}$ is defined by

$$|\psi(t)\rangle_{\rm I} = e^{i\hat{H}_0 t}|\psi(t)\rangle \ . \tag{A.3}$$

Corresponding to this definition, any operator $\hat{O}$ is also translated to the operator $\hat{O}_{\rm I}(t)$ in the interaction picture by the definition

$$\hat{O}_{\rm I}(t) = e^{i\hat{H}_0 t}\hat{O}e^{-i\hat{H}_0 t} \ . \tag{A.4}$$

Since $\hat{H}_0$ is the free field Hamiltonian, the equation of motion of the operator

$$\dot{\hat{O}}_{\rm I}(t) = -i[\hat{O}_{\rm I}(t), \hat{H}_0] \tag{A.5}$$

is fully solvable in terms of the annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ in the same manner as we did for the free fields.

Since the time development of $|\psi(t)\rangle$ is given by $|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)}|\psi(t_0)\rangle$, the time development of $|\psi(t)\rangle_{\rm I}$ is formally expressed as

$$|\psi(t)\rangle_{\rm I} = e^{i\hat{H}_0 t}e^{-i\hat{H}(t-t_0)}|\psi(t_0)\rangle = e^{i\hat{H}_0 t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0 t_0}|\psi(t_0)\rangle_{\rm I} \ , \tag{A.6}$$

that is,

$$|\psi(t)\rangle_{\rm I} = \hat{U}(t,t_0)|\psi(t_0)\rangle_{\rm I} \ : \qquad \hat{U}(t,t_0) \equiv e^{i\hat{H}_0 t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0 t_0} \ . \tag{A.7}$$

Evidently, $\hat{U}(t,t_0)$ is a unitary operator

$$\hat{U}^\dagger(t,t_0) = \hat{U}^{-1}(t,t_0) \tag{A.8}$$

and satisfies the relations

$$\hat{U}(t,t')\hat{U}(t',t_0) = \hat{U}(t,t_0) \ , \quad \hat{U}(t_0,t_0) = \hat{1} \ , \quad \hat{U}(t_0,t) = \hat{U}^{-1}(t,t_0) \ . \tag{A.9}$$

The task left to us is to build up the operator $\hat{U}(t,t_0)$ in a tractable form.

A.2 Time Development Operator

Let us derive the equation of motion of $|\psi(t)\rangle_{\rm I}$ based on the Schrödinger equation (A.1) ;

$$\begin{aligned} i\frac{d}{dt}|\psi(t)\rangle_{\rm I} &= i\frac{d}{dt}\left(e^{i\hat{H}_0 t}|\psi(t)\rangle\right) = e^{i\hat{H}_0 t}\left(-\hat{H}_0 + i\frac{d}{dt}\right)|\psi(t)\rangle \\ &= e^{i\hat{H}_0 t}\left(-\hat{H}_0 + \hat{H}\right)|\psi(t)\rangle = e^{i\hat{H}_0 t}\hat{V}|\psi(t)\rangle \ . \end{aligned} \tag{A.10}$$

Therefore, $|\psi(t)\rangle_{\rm I}$ satisfies

$$i\frac{d}{dt}|\psi(t)\rangle_{\rm I} = \hat{V}_{\rm I}(t)|\psi(t)\rangle_{\rm I} \ . \tag{A.11}$$

Integrating this equation from t_0 to t, we obtain a recursive equation

$$|\psi(t)\rangle_{\rm I} = |\psi(t_0)\rangle_{\rm I} - i\int_{t_0}^{t} dt_1 \hat{V}_{\rm I}(t_1)|\psi(t_1)\rangle_{\rm I} \ . \tag{A.12}$$

Substituting $|\psi(t_1)\rangle_{\rm I}$ in the right-hand-side by the total of this expression and making the same procedure iteratively, we arrive at the expression

$$\begin{aligned} \hat{U}(t,t_0) = \hat{1} + (-i)\int_{t_0}^{t} dt_1 \hat{V}_{\rm I}(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \hat{V}_{\rm I}(t_1)\int_{t_0}^{t_1} dt_2 \hat{V}_{\rm I}(t_2) \\ +(-i)^3\int_{t_0}^{t} dt_1 \hat{V}_{\rm I}(t_1)\int_{t_0}^{t_1} dt_2 \hat{V}_{\rm I}(t_2)\int_{t_0}^{t_2} dt_3 \hat{V}_{\rm I}(t_3) + \cdots \ . \end{aligned} \tag{A.13}$$

Notice that every $\hat{V}_{\rm I}(t_i)$ in each term stands from right to left along the ordering of its time t_i as far as the condition $t \geq t_0$ is satisfied. It is just the T-product. Now we give the identity

$$\int_{t_0}^{t} dt_1 \hat{V}_{\rm I}(t_1)\cdots\int_{t_0}^{t_{N-1}} dt_N \hat{V}_{\rm I}(t_N) = \frac{1}{N!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_N T[\hat{V}_{\rm I}(t_1)\cdots\hat{V}_{\rm I}(t_N)] = \frac{1}{N!}T\left[\int_{t_0}^{t} dt' \hat{V}_{\rm I}(t')\right]^N \tag{A.14}$$

which holds for $t \geq t_0$. We can easily confirm this identity from the definition of the T-product

$$T[\hat{V}_{\rm I}(t_1)\cdots\hat{V}_{\rm I}(t_N)] = \sum_{\{\sigma_i\}}\theta(t_{\sigma_1}-t_{\sigma_2})\cdots\theta(t_{\sigma_{N-1}}-t_{\sigma_N})\hat{V}_{\rm I}(t_{\sigma_1})\cdots\hat{V}_{\rm I}(t_{\sigma_N}) \ , \tag{A.15}$$

where the sum $\{\sigma_i\}$ is taken over all $N!$ permutations of $\{1,\cdots,N\}$. Each term of (A.15), integrated over dt_i's, gives the same result which coincides with the left-hand-side of (A.14), and $N!$ is canceled. Thus, the operator $\hat{U}(t,t_0)$ for $t \geq t_0$ is expressed as

$$\hat{U}(t,t_0) = \sum_{N=0}^{\infty}\frac{(-i)^N}{N!}T\left[\int_{t_0}^{t} dt' \hat{V}_{\rm I}(t')\right]^N = T\left[\exp -i\int_{t_0}^{t} dt' \hat{V}_{\rm I}(t')\right] \qquad t \geq t_0 \ . \tag{A.16}$$

A.3 Green's Function

Now, we are ready to represent the Green's function $G(x_1,\cdots,x_N) = \langle 0|T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)]|0\rangle$ in the framework of the interaction picture. We assume for a while the ordering of the times x_i^0 is $x_1^0 \geq x_2^0 \geq \cdots \geq x_N^0$. From the definition of the Heisenberg operator, we find

$$\begin{aligned} \hat{\phi}(x_1)\hat{\phi}(x_2) &= e^{i\hat{H}x_1^0}\hat{\phi}(\mathbf{x}_1)e^{-i\hat{H}x_1^0}e^{i\hat{H}x_2^0}\hat{\phi}(\mathbf{x}_2)e^{-i\hat{H}x_2^0} \\ &= e^{i\hat{H}x_1^0}e^{-i\hat{H}_0x_1^0}\hat{\phi}_{\rm I}(x_1)e^{i\hat{H}_0x_1^0}e^{-i\hat{H}(x_1^0-x_2^0)}e^{-i\hat{H}_0x_2^0}\hat{\phi}_{\rm I}(x_2)e^{i\hat{H}_0x_2^0}e^{-i\hat{H}x_2^0} \\ &= e^{i\hat{H}x_1^0}e^{-i\hat{H}_0x_1^0}\hat{\phi}_{\rm I}(x_1)\hat{U}(x_1^0,x_2^0)\hat{\phi}_{\rm I}(x_2)e^{i\hat{H}_0x_2^0}e^{-i\hat{H}x_2^0} \ . \end{aligned} \tag{A.17}$$

Therefore, we obtain for $T > x_1^0 \geq x_2^0 \geq \cdots \geq x_N^0 > -T$

$$\begin{aligned}
&\langle 0|\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)|0\rangle \\
&= \langle 0|e^{i\hat{H}x_1^0}e^{-i\hat{H}_0x_1^0}\hat{\phi}_{\rm I}(x_1)\hat{U}(x_1^0,x_2^0)\hat{\phi}_{\rm I}(x_2)\cdots\hat{\phi}(x_{N-1})\hat{U}(x_{N-1}^0,x_N^0)\hat{\phi}_{\rm I}(x_N)e^{i\hat{H}_0x_N^0}e^{-i\hat{H}x_N^0}|0\rangle \\
&= \langle \Omega_T|\hat{U}(T,x_1^0)\hat{\phi}_{\rm I}(x_1)\hat{U}(x_1^0,x_2^0)\hat{\phi}_{\rm I}(x_2)\cdots\hat{\phi}(x_{N-1})\hat{U}(x_{N-1}^0,x_N^0)\hat{\phi}_{\rm I}(x_N)\hat{U}(x_N^0,-T)|\Omega_{-T}\rangle \\
&= \langle \Omega_T|T\left[\hat{\phi}_{\rm I}(x_2)\cdots\hat{\phi}_{\rm I}(x_N)\exp -i\int_{-T}^{T}dt\hat{V}_{\rm I}(t)\right]|\Omega_{-T}\rangle\ ,
\end{aligned} \tag{A.18}$$

where we have defined

$$\langle\Omega_T| = \langle 0|e^{i\hat{H}T}e^{-i\hat{H}_0T} = \langle 0|e^{-i\hat{H}_0T}\ , \qquad |\Omega_{-T}\rangle = e^{-i\hat{H}_0T}e^{i\hat{H}T}|0\rangle = e^{-i\hat{H}_0T}|0\rangle\ . \tag{A.19}$$

Removing the constraint on the ordering of t_i, we obtain the expression for the Green's function

$$\begin{aligned}
G(x_1,\cdots,x_N) &= \langle 0|T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_N)]|0\rangle \\
&= \lim_{T\to+\infty}\langle\Omega_T|T\left[\hat{\phi}_{\rm I}(x_2)\cdots\hat{\phi}_{\rm I}(x_N)\exp -i\int_{-T}^{T}dt\hat{V}_{\rm I}(t)\right]|\Omega_{-T}\rangle\ .
\end{aligned} \tag{A.20}$$

In order to bring this expression to the calculable form, we need some trick. Let us introduce the eigenvectors of the free Hamiltonian $\hat{H}_0$ by

$$\hat{H}_0|n\rangle_{\rm I} = E_n|n\rangle_{\rm I} \qquad\qquad n = 0,1,2,\cdots\ , \tag{A.21}$$

and expand $\langle\Omega_T|$ and $|\Omega_{-T}\rangle$ as

$$\langle\Omega_T| = \sum_{n=0}^{\infty}e^{-iE_nT}\langle 0|n\rangle_{\rm I}\ {}_{\rm I}\langle n|\ , \qquad |\Omega_{-T}\rangle = \sum_{n=0}^{\infty}|n\rangle_{\rm I}\ {}_{\rm I}\langle n|0\rangle e^{-iE_nT}\ . \tag{A.22}$$

Now we replace T in (A.20) with slightly modified value $T(1-i\epsilon)$ with an infinitesimal positive ϵ. Then, the dominant term in $\langle\Omega_T|$ and $|\Omega_{-T}\rangle$ which survives in the limit $T\to+\infty$ is the ground state $|0\rangle_{\rm I}$;

$$\langle\Omega_T| \simeq e^{-iE_0T(1-i\epsilon)}\langle 0|0\rangle_{\rm I}\ {}_{\rm I}\langle 0|\ , \qquad |\Omega_{-T}\rangle \simeq |0\rangle_{\rm I}\ {}_{\rm I}\langle 0|0\rangle e^{-iE_0T(1-i\epsilon)}\ . \tag{A.23}$$

On the other hand, the normalization of the vacuum $\langle 0|0\rangle = 1$ gives

$$1 = \langle\Omega_T|T\left[\exp -i\int_{-T}^{T}dt\hat{V}_{\rm I}(t)\right]|\Omega_{-T}\rangle\ . \tag{A.24}$$

This determines the value of the product of the coefficients of (A.23) as

$$e^{-i2E_0T(1-i\epsilon)}\langle 0|0\rangle_{\rm I}\ {}_{\rm I}\langle 0|0\rangle \simeq \frac{1}{{}_{\rm I}\langle 0|T\left[\exp -i\int_{-T(1-i\epsilon)}^{T(1-i\epsilon)}dt\hat{V}_{\rm I}(t)\right]|0\rangle_{\rm I}}\ . \tag{A.25}$$

Thus, we arrive at the expression

$$G(x_1,\cdots,x_N) = \frac{{}_{\rm I}\langle 0|T\left[\hat{\phi}_{\rm I}(x_2)\cdots\hat{\phi}_{\rm I}(x_N)\exp -i\int_{-\infty(1-i\epsilon)}^{+\infty(1-i\epsilon)}dt\hat{V}_{\rm I}(t)\right]|0\rangle_{\rm I}}{{}_{\rm I}\langle 0|T\left[\exp -i\int_{-\infty(1-i\epsilon)}^{+\infty(1-i\epsilon)}dt\hat{V}_{\rm I}(t)\right]|0\rangle_{\rm I}}\ . \tag{A.26}$$

In the most cases, the interaction does not contain the derivatives of the fields. It means

$$L = L_0 + L_{\rm int}\ \Rightarrow\ \hat{H} = \hat{H}_0 + \hat{V} = \hat{H}_0 - \hat{L}_{\rm int}\ . \tag{A.27}$$

Therefore, replacing $\int dt\ \hat{V}_{\rm I}(t)$ in (A.26) by $-\int d^4x\ \mathcal{L}_{\rm int}(\hat{\phi}_{\rm I}(x))$, we obtain the formula

$$G(x_1,\cdots,x_N) = \frac{{}_{\rm I}\langle 0|T\left[\hat{\phi}_{\rm I}(x_2)\cdots\hat{\phi}_{\rm I}(x_N)\exp i\int d^4x\ \mathcal{L}_{\rm int}(\hat{\phi}_{\rm I}(x))\right]|0\rangle_{\rm I}}{{}_{\rm I}\langle 0|T\left[\exp i\int d^4x\ \mathcal{L}_{\rm int}(\hat{\phi}_{\rm I}(x))\right]|0\rangle_{\rm I}}\ . \tag{A.28}$$

A.4 Generating Functional

Let us define the generating functional of Green's function by

$$Z[J] = {}_{\rm I}\langle 0|T\left[\exp i\int d^4x\ \left(\mathcal{L}_{\rm int}(\hat{\phi}_{\rm I}(x)) + J(x)\hat{\phi}_{\rm I}(x)\right)\right]|0\rangle_{\rm I}$$

$$= \exp i\int d^4x\ \mathcal{L}_{\rm int}\left(\frac{\delta}{i\delta J(x)}\right) Z_0[J]\ , \tag{A.29}$$

$$Z_0[J] = {}_{\rm I}\langle 0|T\left[\exp i\int d^4x\ J(x)\hat{\phi}_{\rm I}(x)\right]|0\rangle_{\rm I}\ . \tag{A.30}$$

Since $Z_0[J]$ is a free field generating functional, it is explicitly calculable. It precisely coincides to the $Z_0[J]$ we have calculated in the path integral method. Originary, the Wick's expansion theorem was invented to calculate such quantities in the operator formalism. Here, we take another approach and calculate

$$\frac{\delta}{i\delta J(x_1)}Z_0[J] = {}_{\rm I}\langle 0|T\left[\hat{\phi}_{\rm I}(x_1)\exp i\int d^4x\ J(x)\hat{\phi}_{\rm I}(x)\right]|0\rangle_{\rm I}$$

$$= {}_{\rm I}\langle 0|T\left[\hat{\phi}_{\rm I}(x_1)\sum_{M=0}^{\infty}\frac{i^M}{M!}\left(\int d^4x\ J(x)\hat{\phi}_{\rm I}(x)\right)^M\right]|0\rangle_{\rm I}\ . \tag{A.31}$$

Since $\hat{\phi}_{\rm I}(x_1)$ satisfies the free field equation of motion, it must be contracted to one of the M $\hat{\phi}_{\rm I}(x)$'s. Thus, (A.31) is expressed as

$$\frac{\delta}{i\delta J(x_1)}Z_0[J] = {}_{\rm I}\langle 0|T\left[\hat{\phi}_{\rm I}(x_1)\ i\int d^4x\ J(x)\hat{\phi}_{\rm I}(x)\right]|0\rangle_{\rm I}$$

$$\times\sum_{M=1}^{\infty}\frac{i^{(M-1)}}{(M-1)!}\ {}_{\rm I}\langle 0|T\left[\left(\int d^4x\ J(x)\hat{\phi}_{\rm I}(x)\right)^{(M-1)}\right]|0\rangle_{\rm I}$$

$$= -\int d^4x\ \Delta_F(x_1-x)J(x)Z_0[J]\ . \tag{A.32}$$

The integration of this equation gives the correct answer

$$Z_0[J] = \exp -\frac{1}{2}\int d^4xd^4yJ(x)i\Delta_F(x-y)J(y)\ . \tag{A.33}$$

Now that we have obtained $Z[J]$, we can go back to section 5.3 and proceed further in the same way.